A β-Factor for Stability: Provably Stable, High-Order Splitting Schemes for Incompressible Flows

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ABSTRACT

The numerical simulation of incompressible fluid dynamics, governed by the Navier-Stokes equations (NSEs), presents persistent challenges related to computational efficiency and numerical stability, especially for higher-order accurate methods. This paper introduces and analyzes a new class of fully decoupled, higher-order consistent splitting schemes for the incompressible Navier-Stokes equations. The core of our methodology is the formulation of schemes based on Taylor series expansions at a future time point $tn+\beta$, where $\beta \geq 1$ is a selectable free parameter. This approach generalizes the classical Backward Differentiation Formula (BDF) methods. The primary contribution of this work is a rigorous stability and error analysis for these schemes. We demonstrate that by selecting appropriate values for the parameter β—specifically, β=3 for the second-order scheme, β=6 for the third-order scheme, and β=9 for the fourth-order scheme—the resulting numerical solutions are uniformly bounded in a strong norm. This constitutes a proof of unconditional stability. Furthermore, we establish optimal global-in-time convergence rates for these schemes in both two and three-dimensional domains. To the best of our knowledge, these findings represent the first comprehensive stability and convergence results for any fully decoupled scheme for the Navier-Stokes equations with an order of accuracy higher than two. The theoretical analysis is substantiated by numerical experiments, which validate the unconditional stability of the new third- and fourth-order schemes. In contrast, we show that schemes based on the conventional BDF approach (i.e., β=1) are not unconditionally stable. The proposed schemes achieve their theoretically predicted orders of convergence, offering a robust and efficient pathway for high-fidelity simulations of incompressible flows..

Keywords: Navier-Stokes Equations, Consistent Splitting Schemes, Higher-Order Methods, Stability Analysis, Error Analysis, Computational Fluid Dynamics, Decoupled Methods.

1. Introduction

1.1. Broad Background and Historical Context

The Navier-Stokes equations (NSEs) are the cornerstone of fluid dynamics, providing a mathematical description of the motion of viscous, incompressible fluids [11, 13, 15]. Their applications are vast and critical, spanning numerous fields of science and engineering, from aerospace design and weather forecasting to biomedical flows and industrial processing [16]. The equations are a system of nonlinear partial differential equations that couple the fluid velocity $\mathbf{u}$ and pressure p. A key feature is the incompressibility constraint, which mandates that the velocity field must remain divergence-free at all points in the fluid domain [29]. This constraint, along with the inherent nonlinearity of the convective term, poses significant challenges for the development of accurate and efficient numerical solution strategies. Due to the immense practical importance of the NSEs, an enormous body of research has been dedicated to their numerical approximation [17]. The goal is to develop schemes that are not only accurate but also computationally efficient and robustly stable, particularly for long-time simulations or flows at low viscosity (high Reynolds number), where complex, multi-scale phenomena can emerge.

1.2. Critical Literature Revie

Numerical methods for the incompressible NSEs can be broadly categorized into two families: coupled approaches and decoupled approaches [9]. Coupled methods [2, 6, 7] solve for the velocity and pressure unknowns simultaneously within a large, monolithic system of equations at each time step. While this approach can be very robust and avoids certain types of splitting errors, it leads to large, ill-conditioned linear systems that can be computationally prohibitive to solve, especially for large-scale, three-dimensional problems. Consequently, decoupled approaches have gained widespread popularity due to their superior computational efficiency [9]. These methods, often termed splitting schemes, break down the complex coupled problem into a sequence of smaller, simpler sub-problems at each time step. This family includes projection-type methods [4, 8, 10, 11, 12, 19, 23, 24, 25, 26, 30, 31] and consistent splitting methods [13, 18, 28, 32].

Projection methods, first introduced in the 1960s, are perhaps the most widely used. However, a well-known deficiency of most projection-type schemes is the introduction of a fundamental splitting error that arises from the inconsistent treatment of the pressure gradient and the viscous term. This error pollutes the pressure approximation and prevents the velocity from achieving its full order of accuracy in strong norms [19]. To address this accuracy limitation, consistent splitting schemes were developed [13, 18]. These schemes are carefully designed to eliminate the leading-order splitting error, thereby allowing for the possibility of full-order accuracy. The gauge method is another related approach that also seeks to improve accuracy and stability [5, 22].

While first-order consistent splitting schemes are well-established, developing stable higher-order versions has proven to be exceptionally difficult. This leads to what has been a long-standing open question in the field [20]: how to construct unconditionally stable, fully decoupled schemes of second-order or higher, complete with a rigorous stability and error analysis. While many high-order splitting methods have been proposed [19, 32], their stability often relies on severe time-step restrictions, or their stability has not been rigorously proven.

The stability analysis of higher-order multi-step methods, such as the Backward Differentiation Formula (BDF) schemes, has a rich history in the context of parabolic equations. The seminal work by Nevanlinna and Odeh [21] introduced powerful multiplier techniques, based on Dahlquist's G-stability theory [3], to prove the stability of BDF methods. This energy-based technique was later extended to the six-step BDF method [1]. Our recent work has built upon this foundation, creating a new class of generalized BDF schemes for parabolic equations based on Taylor expansions at a future time point, $t_{n+\beta}$, and establishing their stability properties [17]. We also recently introduced a second-order consistent splitting scheme for the NSEs and provided a rigorous stability and error analysis [16]. This paper represents a significant advancement over that work.

1.3. The Identified Research Gap

Despite decades of research, the literature lacks a rigorous framework for constructing and analyzing unconditionally stable, fully decoupled consistent splitting schemes for the NSEs that are of third-order or higher accuracy [7, 20]. While such schemes can be formally constructed, their stability analysis remains an unresolved major challenge. The primary difficulty lies in controlling the error introduced by the explicit treatment of the pressure term in the momentum equation when using a higher-order time-stepping formula. Standard high-order BDF schemes (corresponding to $\beta=1$ in our new framework) are known to be unstable without stringent time-step constraints, as our numerical results will confirm [8].

1.4. Study Rationale, Objectives, and Hypotheses

The central rationale of this study is to fill this critical gap by developing a new class of high-order consistent splitting schemes that are provably and unconditionally stable. We aim to provide a comprehensive theoretical framework and practical numerical methods that are both accurate and efficient. The primary objectives of this paper are:

- To construct a new class of fully decoupled, consistent splitting schemes for the Navier-Stokes equations, achieving second-, third-, and fourth-order temporal accuracy, based on a generalized BDF framework [5, 17].
- To perform a rigorous and complete stability and error analysis for these new schemes, proving that by selecting specific values for the parameter β (namely $\beta_2=3, \beta_3=6, \beta_4=9$), the schemes are unconditionally stable.
- To establish global-in-time optimal error estimates for the proposed schemes, demonstrating that they achieve their formal order of accuracy in both 2D and 3D domains [6].
- To validate the theoretical findings through numerical experiments, comparing the stability and accuracy of the new schemes against their classical BDF counterparts.

We hypothesize that the proposed schemes, with their specific choices of β, will be unconditionally stable, in stark contrast to the conditional stability of schemes based on the standard BDF formulation ($\beta=1$). Furthermore, we hypothesize that the schemes will exhibit their respective theoretical orders of convergence in numerical tests, confirming the correctness of our error analysis.

2. Methods

2.1. Research Design

The research methodology employed in this study is a combination of theoretical analysis and numerical validation. The core of the work is the mathematical construction and rigorous analysis of a new family of numerical schemes. The investigation proceeds in a structured manner:

1. We first construct generalized Backward Differentiation Formula (BDF) consistent splitting schemes of order k (where $k=2,3,4$) for the linear, time-dependent Stokes equations.
2. We then extend these schemes to the complete, nonlinear Navier-Stokes equations using an implicit-explicit (IMEX) treatment.
3. A rigorous error analysis is conducted for the schemes applied to the full NSEs to prove optimal-order, global-in-time convergence.

4. Finally, a series of numerical experiments are designed to validate the theoretical results.

2.2. Preliminaries and Governing Equations

The physical system under consideration is that of an incompressible, viscous fluid within a bounded domain $\Omega\subset R^d$ (for $d=2,3$). The motion of this fluid is described by the incompressible Navier-Stokes equations [11, 13]:

$$\partial t\partial u+u\cdot\nabla u-\nu\Delta u+\nabla p=f(1.1a)\nabla\cdot u=0(1.1b)$$

Here, **u(x,t)** is the fluid velocity vector, *p(x,t)* is the kinematic pressure, $\nu>0$ is the constant kinematic viscosity, and **f(x,t)** is an external body force. These equations are supplemented with a suitable initial condition and a no-slip boundary condition **u**=0 on the domain boundary $\partial\Omega$.

To facilitate the analysis, we introduce standard notations for function spaces. Let $L^p(\Omega)$ and $H^k(\Omega)$ denote the usual Lebesgue and Sobolev spaces. We define the solenoidal (divergence-free) function space $\mathbf{V}=\{\mathbf{v}\in H_0^1(\Omega):\nabla\cdot\mathbf{v}=0\}$. The trilinear form arising from the convective term is defined as $b(\mathbf{u},\mathbf{v},\mathbf{w})=\int_\Omega(u\cdot\nabla)\mathbf{v}\cdot\mathbf{w}dx$, which satisfies various inequalities crucial for the analysis [29]. Our analysis will also rely on the discrete version of Gronwall's Lemma [14].

A cornerstone of our stability proof is the concept of the Stokes pressure and a related commutator estimate developed by Liu, Liu, and Pego [20]. For a vector field $\mathbf{u}\in H^2(\Omega)$, the Stokes pressure $p_s(\mathbf{u})$ is defined via:

$$\nabla p_s(u)=(\Delta P-P\Delta)u(2.3)$$

where P is the Leray-Helmholtz projection operator. It was proven in [20] that this operator satisfies the critical estimate for any $\varepsilon > 0$:

$$\int_\Omega|(\Delta P-P\Delta)u|^2\le(21+\epsilon)\int_\Omega|\Delta u|^2+C\int_\Omega|\nabla u|^2(2.5)$$

This result is valid under the condition that Ω has a C^3 boundary and is essential for controlling the explicit pressure term in our decoupled scheme. Finally, our stability analysis leverages the powerful G-stability theory of Dahlquist [3]

2.3. Construction of the Generalized BDF Consistent Splitting Schemes

Following our prior work on parabolic equations [17], we construct generalized k-th order BDF-type schemes by employing a Taylor series expansion of a function $\phi(t)$ around the future time point $t_{n+\beta}=(n+\beta)\delta t$. This leads to discrete operators which we denote as:

$$A_{k}^{\beta}(\phi^{i})=\sum_{q=0}^{k}a_{k,q}(\beta)\phi^{i-k+q}$$$$B_{k}^{\beta}(\phi^{i})=\sum_{q=0}^{k-1}b_{k,q}(\beta)\phi^{i-k+1+q}$$$$C_{k}^{\beta}(\phi^{i})=\sum_{q=0}^{k-1}c_{k,q}(\beta)\phi^{i-k+1+q}$$

The coefficients—$a_{k,q}(\beta),b_{k,q}(\beta)$, and $c_{k,q}(\beta)$—are uniquely determined by solving systems of linear equations that enforce k-th order accuracy. Using these operators, the proposed k-th order generalized BDF consistent splitting scheme for the time-dependent Stokes equation is formulated as a two-stage process at each time step *n*:

- **Momentum Step:** Solve for the velocity u^{n+1}:$\delta t1 A_k^\beta(u^{n+1})-\nu\Delta B_k^\beta(u^{n+1})+\nabla C_k^\beta(p^n)=0(3.9a)$
- **Pressure-Correction Step:** Solve for the pressure p^{n+1}:$(\nabla p^{n+1},\nabla q)=-\nu(\nabla\times\nabla\times u^{n+1},\nabla q),\forall q\in H^1(\Omega)(3.9b)$

For the full nonlinear Navier-Stokes equations, we introduce a BDF-IMEX scheme where the nonlinear convective term is treated explicitly:

- **Momentum Step:**$\delta t1A_k^{\beta_k}(u^{n+1})-\nu\Delta B_k^{\beta_k}(u^{n+1})+\nabla C_k^{\beta_k}(p^n)+C_k^{\beta_k}(u^n)\cdot\nabla C_k^{\beta_k}(u^n)=f^{n+\beta_k}$ (4.1a)
- **Pressure-Correction Step:**$(\nabla p^{n+1},\nabla q)=(f^{n+1}-u^{n+1}\cdot\nabla u^{n+1}-\nu\nabla\times\nabla\times u^{n+1},\nabla q),\forall q\in H^1(\Omega)(4.1b)$

2.4. Stability Analysis Protocol

The stability analysis is the technical core of this paper. Our analysis shows that unconditional stability is not guaranteed for arbitrary choices of β. However, we have identified specific values that do confer this property for schemes of order two, three, and four:

$$\beta_2=3,\beta_3=6,\beta_4=9(3.12)$$

A crucial and non-trivial step in the analysis is a delicate splitting of the implicitly treated viscous term operator, $B_k^{\beta_k}$. We decompose it as:

$$B_k^{\beta_k}(u^{n+1})=\eta_k C_k^{\beta_k}(u^{n+1})+D_k^{\beta_k}(u^{n+1})+F_k^{\beta_k}(u^{n+1})(3.16)$$

Here, η_k is a positive constant, and $D_k^{\beta_k}$ and $F_k^{\beta_k}$ are new linear operators. The parameter η_k must be chosen such that $\eta_k>22\approx0.7071$; we use $\eta_k=0.71$. The stability proof relies on two pivotal lemmas (Lemma 3.1 and 3.2) concerning the properties of polynomials whose coefficients are derived from the operators $A_k^{\beta_k},C_k^{\beta_k}$, and the newly defined $D_k^{\beta_k}$. These lemmas establish that specific ratios of these polynomials satisfy the conditions of Dahlquist's G-stability theory [3], a result adapted from multiplier techniques [17, 21]. The overall proof strategy for establishing unconditional stability (Theorem 3.3) for the Stokes problem is as follows:

1. Take the L^2 inner product of the momentum equation (3.9a) with the test function $-\Delta C_k^{\beta_k}(u^{n+1})$.
2. Apply the viscous term splitting (3.16).
3. Bound the explicit pressure term, $(\nabla C_k^{\beta_k}(p^n),-\Delta C_k^{\beta_k}(u^{n+1}))$, using the Cauchy-Schwarz inequality and the

crucial Stokes pressure commutator estimate from [20].

4. Combining these estimates and summing over time steps yields a uniform energy bound, proving unconditional stability.

2.5. Error Analysis Plan

The error analysis for the full nonlinear scheme (Theorem 4.1) is performed via a mathematical induction argument. We assume that a uniform bound on the solution's gradient, $\|\nabla u^i\| \le C_0$, holds for all time steps up to $i=n$, and then prove it must also hold for step $n+1$. The proof proceeds in three main steps:

1. **Uniform Bound on Numerical Solution:** First, we establish an a priori bound on the $L^\infty(H^1)\cap L^2(H^2)$ norm of the numerical solution u^n, assuming the induction hypothesis.
2. **Velocity Error Estimate:** We derive the error equation which governs the evolution of the error $e^n=u^n-u(t_n)$: $A_k^{\beta_k}(e^{i+1})-\delta t\Delta B_k^{\beta_k}(e^{i+1})+\delta t\nabla C_k^{\beta_k}(e_p^i)+\cdots=\delta t P_k^i+\delta t Q_k^i+R_k^i+\delta t S_k^i$ (4.24)
   The right-hand side consists of truncation and consistency errors. We then perform an energy analysis on this error equation, similar to the stability proof. The nonlinear error terms are carefully estimated using the induction hypothesis and Sobolev inequalities from [29]. Applying the discrete Gronwall lemma [14] yields the desired optimal-order error bound.
3. **Pressure Error Estimate:** Finally, an error bound for the pressure is derived by analyzing the error equation for the pressure-correction step, again using the Stokes pressure properties [20] and the previously established velocity error bounds.

3. Results

This section presents the main theoretical achievements of this study concerning the stability and convergence of the new schemes, followed by a summary of the numerical experiments conducted to validate these theories.

3.1. Theoretical Results: Stability and Convergence

The analytical investigation yielded two main theorems that establish the favorable properties of the proposed class of schemes.

- **Theorem 3.3 (Unconditional Stability for Stokes Equations):** For the time-dependent Stokes problem, the k-th order consistent splitting scheme (3.9), with the parameter β chosen as $\beta_k\in\{3,6,9\}$ for $k=2,3,4$ respectively, is unconditionally stable. Specifically, the solution is uniformly bounded in the $L^\infty(H^1)\cap L^2(H^2)$ norm for any time step size $\delta t>0$. This result is significant as it provides the first proof of unconditional stability for any fully decoupled scheme for the time-dependent Stokes equations with an order of accuracy of three or higher.
- Theorem 4.1 (Optimal Error Estimates for Navier-Stokes Equations): Let the solution of the Navier-Stokes equations (1.1) be sufficiently smooth. Then the solution of the k-th order BDF-IMEX scheme (4.1) with $\beta_k\in\{3,6,9\}$ for $k=2,3,4$ converges to the exact solution with an optimal global-in-time error estimate. For a sufficiently small time step δt, the following bound holds for all $n+1\le T/\delta t$: $\|\nabla e^{n+1}\|^2+\delta t\sum_{i=0}^{n+1}(\|\Delta e^i\|^2+\|\nabla e_p^i\|^2)\le C\delta t^{2k}$ (4.3)

  where e^n and e_p^n are the errors in velocity and pressure, respectively, and the constant C is independent of δt. This theorem establishes that the schemes achieve their formal order of accuracy, k, and represents the first such convergence result for any fully decoupled, higher-than-second-order scheme for the full Navier-Stokes equations.

3.2. Numerical Validation 1: Stability Comparison

To test the theoretical stability claims, we implemented the third- and fourth-order schemes to solve the full Navier-Stokes equations with a small viscosity $\nu=0.005$. We compared the performance of our new schemes (with $\beta_3=6$ and $\beta_4=9$) against the corresponding standard BDF3 and BDF4 schemes ($\beta=1$). The results were unequivocal. The standard third- and fourth-order BDF schemes proved to be unstable for even moderately small time steps. In stark contrast, our new schemes were perfectly stable and produced accurate energy evolution profiles for a much larger time step of $\delta t=0.05$, demonstrating their unconditional stability in practice.

3.3. Numerical Validation 2: Convergence Rates

To verify the accuracy and convergence rates predicted by Theorem 4.1, we conducted a second set of experiments using a problem with a known analytical solution. We tested the second-order scheme with $\beta_2=3$, the third-order scheme with $\beta_3=6$, and the fourth-order scheme with $\beta_4=9$. The results showed that the slopes of the error decay on a log-log plot were approximately 2, 3, and 4 for the respective schemes. This confirms that the new schemes achieve their expected theoretical orders of convergence in practice, validating the error analysis of Theorem 4.1.

4. Discussion

4.1. Interpretation of Key Findings

The central achievement of this research is the successful development and rigorous analysis of a new class of high-order, fully decoupled, and unconditionally stable schemes for the incompressible Navier-Stokes equations [7]. This work effectively resolves a long-standing open problem in computational fluid dynamics [20]. The success of our approach hinges on a synergistic combination of three key ideas: the generalization of BDF methods via a Taylor expansion around a future time point $t_{n+\beta}$; the judicious selection of specific β values; and the novel splitting of the implicit viscous term (3.16). This framework successfully tames the instability that typically arises from the explicit treatment of the pressure term [18]. The explicit pressure is ultimately controlled by leveraging the sophisticated Stokes commutator estimate developed by Liu, Liu, and Pego [20].

4.2. Comparison with Previous Literature

The contributions of this paper should be viewed in the context of decades of research on numerical methods for the NSEs.

- **Versus Projection Methods:** Our work offers a significant advantage over commonly used projection methods [4, 9, 25]. As "consistent splitting" schemes [13], our methods are designed to be free from the leading-order splitting error that fundamentally limits the accuracy of standard projection schemes [19].
- **Versus Other High-Order Schemes:** While numerous high-order methods for the NSEs have been proposed [19, 32], they often are either coupled schemes that lead to computationally expensive linear systems [6, 7], or they are decoupled schemes that lack a rigorous proof of unconditional stability. Our work provides this missing piece.
- **Versus Standard BDF Schemes:** The numerical results presented herein provide a stark demonstration of the superiority of our approach over methods based on the standard high-order BDF formulas (β=1). The instability of the BDF3 and BDF4 schemes shown in our tests is a well-known practical issue.
- **Relation to Prior Work:** This paper is a direct and highly non-trivial extension of our own previous research. It improves upon the second-order scheme presented in [16] and applies the general theoretical framework we developed for parabolic-type equations in [17] to the much more intricate Navier-Stokes system. The successful adaptation of G-stability theory [3] and multiplier methods [21] to this new context is a core technical contribution.

4.3. Strengths and Limitations of the Study

The primary strength of this work lies in its novelty and rigor, providing the first-of-their-kind stability and convergence proofs for fully decoupled, consistent splitting schemes for the NSEs with an order of accuracy higher than two [7]. However, the study also has several limitations that point toward avenues for future research. The analysis presented is for semi-discrete schemes, where only time is discretized. The stability proof for the pressure term formally requires the fluid domain Ω to have a smooth boundary of class C^3. While our numerical results demonstrate excellent performance on a simple square domain, extending the rigorous proof to polygonal domains is an open question. The chosen values of β are sufficient for stability, but they may not be the smallest possible values. The analysis is currently limited to schemes up to the fourth order.

4.4. Implications for Theory and Practice

The implications of this research are twofold.

- **For Theory:** This work resolves a long-standing open question regarding the existence of provably stable, high-order, decoupled numerical schemes for the NSEs [20]. The analytical technique we have developed provides a new paradigm for the stability analysis of complex fluid dynamics problems.
- **For Practice:** Our results provide computational scientists and engineers with a new class of powerful tools for simulating incompressible flows [16]. Their proven high order of accuracy allows for the resolution of fine-scale flow features with fewer degrees of freedom, and their unconditional stability can lead to dramatic computational savings in long-time simulations.

4.5. Conclusion and Future Research Directions

In conclusion, this paper has introduced a new class of higher-order consistent splitting schemes for the incompressible Navier-Stokes equations. By constructing the schemes from a Taylor expansion at a future time point $t_{n+\beta}$ and making specific choices for the parameter β, we have developed the first fully decoupled schemes of third- and fourth-order that are provably unconditionally stable. This work opens up several promising avenues for future investigation, including the extension to even higher orders (fifth and sixth); a comprehensive stability and error analysis of fully discretized schemes; extension of the theory to domains with less regularity; and application of the methodology to coupled multi-physics systems involving the NSEs.

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