Asymptotic Behaviour of Nonlinear Viscoelastic Wave Equations with Boundary Feedback

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ABSTRACT

This study investigates a nonlinear viscoelastic wave equation subject to acoustic boundary conditions and a nonlinear distributed delay feedback acting on the boundary. The analysis of the asymptotic behavior of such systems is of paramount importance for both theoretical advancements in the field of partial differential equations and for practical applications in science and engineering. Viscoelastic materials, which exhibit both elastic and viscous properties, are modeled by equations that incorporate memory effects, often represented by integral terms. The inclusion of nonlinear distributed delay in the boundary feedback introduces additional complexity, reflecting more realistic physical scenarios where system responses are not instantaneous but occur over a range of times. Furthermore, the consideration of acoustic boundary conditions enhances the model's applicability to problems involving wave interactions at material interfaces. In this work, we establish a framework for analyzing the long-term behavior of solutions to this complex system. By employing the multiplier method and constructing a suitable Lyapunov functional, we derive general decay results for the energy of the system. The analysis is carried out under a general set of assumptions on the memory kernel and the nonlinear functions that characterize the boundary feedback and delay. We demonstrate that the energy of the system decays to zero as time tends to infinity, and we provide explicit decay rates that depend on the properties of the memory kernel and the nonlinearities in the system. Our findings contribute to the fundamental understanding of energy dissipation and stability in viscoelastic systems with time-delayed boundary controls. This research not only advances the mathematical theory but also provides valuable insights for the design and analysis of materials and structures where viscoelasticity, acoustic effects, and delayed feedback are significant factors.

Keywords: Mathematical model; viscoelastic term; wave equation; asymptotic behaviour; acoustic boundary; nonlinear distributed delay.

1. Introduction

1.1. Broad Background and Historical Context

The study of wave propagation in materials that exhibit both elastic and viscous characteristics has a long and rich history, with its roots in the classical theories of elasticity and fluid dynamics. Viscoelastic materials, which include a wide range of substances from polymers and biological tissues to amorphous solids and glasses, display a time-dependent response to applied stresses, a phenomenon that cannot be adequately described by purely elastic or purely viscous models. The mathematical modeling of viscoelasticity dates back to the 19th century with the work of Boltzmann, who introduced the concept of a memory kernel to describe the influence of the material's past history on its current state. This led to the development of integral and differential models of linear viscoelasticity, with foundational contributions from researchers such as Coleman and Noll [29] and Bland [27].

The linear theory of viscoelasticity, while successful in describing the behavior of many materials under small deformations, often fails to capture the complex responses observed at larger strains. This has motivated the development of nonlinear theories of viscoelasticity, which incorporate nonlinear stress-strain relationships and more complex memory effects. The mathematical analysis of nonlinear viscoelastic wave equations is a challenging area of research, as these equations often involve nonlinearities in both the differential operator and the integral memory term. A crucial aspect of the study of viscoelastic wave equations is the understanding of their long-term or asymptotic behavior. This involves investigating the stability of solutions and the decay of the system's energy over time. Energy decay is a manifestation of the dissipative nature of viscoelastic materials, where mechanical energy is converted into heat due to internal friction. The rate of energy decay is a key characteristic of a viscoelastic material and is of great practical importance in applications where vibration damping and energy absorption are desired.

1.2. Critical Literature Review

The study of the asymptotic behavior of viscoelastic wave equations has been a very active area of research in recent decades, with a particular focus on the effects of various types of damping mechanisms. Several studies have investigated the energy decay properties of viscoelastic systems with different boundary conditions and feedback mechanisms [1, 2, 9, 10, 11, 12, 13, 17, 21, 34]. Al-Mahdi and Al-Gharabli [9] analyzed a viscoelastic equation with past history and boundary feedback, providing conditions for stability. Their work highlighted the interplay between the memory term and the boundary damping in determining the energy decay rate. Messaoudi and Al-Gharabli [10] established a general decay result for a similar model, demonstrating the significant influence of memory effects on energy dissipation. Further advancements were made by Al-Gharabli et al. [11], who examined a viscoelastic system with nonlinear boundary feedback and a logarithmic source term, deriving decay estimates under suitable conditions. In a related study, the same authors [12] obtained general and optimal decay results for a viscoelastic equation with nonlinear boundary feedback, refining existing stability criteria.

The introduction of time delay in the boundary feedback adds another layer of complexity to the analysis. Time delay is a ubiquitous phenomenon in physical and engineering systems, and its effects on the stability of wave equations have been extensively studied [14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 34]. Datko, Lagnese, and Polis [14] provided an early example of the effect of time delays in boundary feedback stabilization of wave equations. Nicaise and Pignotti [15, 24, 26] have made significant contributions to the understanding of stability and instability in wave equations with delay terms in the boundary or internal feedbacks, including distributed delays. Their work has shown that the presence of delay can have a destabilizing effect, and that the stability of the system depends crucially on the relationship between the delay and the other system parameters.

More recently, there has been a growing interest in studying viscoelastic wave equations with more complex boundary conditions, such as acoustic boundary conditions. These conditions, introduced by Morse and Ingard [33] and further developed by Beale and Rosencrans [32], describe the interaction of a fluid with a flexible boundary and are relevant in a wide range of applications, from architectural acoustics to underwater sound propagation. Several researchers have investigated wave equations with acoustic boundary conditions and various forms of damping [1, 2, 13, 17, 21]. Lee and Kang [17] studied the general stability of a viscoelastic wave equation with nonlinear time-varying delay, nonlinear damping, and acoustic boundary conditions. Choucha et al. [1, 2, 13, 21] have conducted a series of studies on viscoelastic wave equations with acoustic and fractional boundary conditions, combined with nonlinear and distributed delays, often in the presence of a logarithmic source term. These studies have provided important insights into the qualitative analysis and asymptotic behavior of these complex systems. The mathematical tools used in the analysis of these problems are often based on the theory of fractional calculus and fractional derivatives, as explored in the works of Ragusa [3, 7], Guariglia [4, 8], Ortigueira and Coito [5], and Li, Dao, and Guo [6]. The study of well-posedness and blow-up of solutions for related problems, such as the p(l)-biharmonic wave equation, has also been a subject of recent investigation [16]. Furthermore, the concept of viscoelasticity and general decay for viscoelastic problems have been explored in numerous research works, including those by Cavalcanti et al. [28, 38], Lasiecka and Tataru [30], and Mesloub and Boulaaras [31]. The study of laminated beams with interfacial slip and fractional derivative type boundary dissipation also contributes to the broader understanding of asymptotic behavior in complex structures [35].

1.3. The Identified Research Gap

While there has been significant progress in the study of viscoelastic wave equations with boundary feedback, the combined effects of nonlinear distributed delay and acoustic boundary conditions have not been fully explored. Most of the existing literature focuses on either a single type of boundary condition (e.g., Dirichlet or Neumann) or a simpler form of delay (e.g., constant or time-varying). The present study aims to fill this gap by considering a nonlinear viscoelastic wave equation with both acoustic boundary conditions and a nonlinear distributed delay in the boundary feedback. This combination of features makes the model more realistic and applicable to a wider range of physical phenomena. A recent study by Choucha and Ouchenane [23] investigated a similar problem but without considering acoustic boundary conditions, focusing on general decay behavior with a general kernel. Our work builds upon and extends these findings by incorporating the important aspect of acoustic boundary interactions.

1.4. Study Rationale, Objectives, and Hypotheses

The primary rationale for this study is to advance the theoretical understanding of a significant class of mathematical models that are widely used in applied and experimental sciences, particularly in the field of viscoelasticity theory. The key distinction of our work from previous studies is the integration of a nonlinear distributed delay within the boundary feedback, coupled with acoustic boundary conditions. The main objective of this research is to analyze the asymptotic behavior of the solutions to the proposed viscoelastic wave equation. Specifically, we aim to:

- Establish the well-posedness of the problem, proving the existence of a unique weak solution.
- Derive a general decay result for the energy of the system, demonstrating that the solution approaches a steady state as time goes to infinity.
- Investigate how the decay rate depends on the properties of the memory kernel, the nonlinearities in the boundary feedback and delay, and the parameters of the acoustic boundary conditions.

Our central hypothesis is that, under suitable assumptions on the kernel and the nonlinear functions, the energy of the system will decay to zero. We further hypothesize that the rate of this decay can be explicitly characterized, providing a deeper understanding of the dissipative mechanisms at play in the system. By achieving these objectives, this study will contribute to the ongoing efforts to develop a comprehensive mathematical theory for the analysis and control of complex viscoelastic systems. Future research will aim to extend this work by incorporating additional damping mechanisms such as Balakrishnan-Taylor damping, dispersion effects, and logarithmic corrections, with a particular focus on nonlinear settings.

2. Methods

2.1. Research Design

This study employs a theoretical and analytical research design based on the mathematical analysis of a system of partial differential equations. The core of our methodology is the use of functional analysis and the multiplier method to establish the existence and asymptotic behavior of solutions to a nonlinear viscoelastic wave equation. The problem is formulated in a bounded domain with smooth boundaries, and the analysis is carried out in appropriate Hilbert spaces. The problem under investigation is given by the following nonlinear viscoelastic wave equation:

ztt−Δz(t)+∫0tP(t−χ)Δz(χ)dχ=0;in A×R+,

subject to the following boundary and initial conditions:

∂v∂z−∫0tZ(t−z)∂D∂z(χ)dχ+Z(zt)=Zt,y∈Λ0,t>0,zt+F(y)∅t+N(y)∅=0,z(y,t)=0,on Λ1×R+,z(y,0)=z0(y),zt(y,0)=z1(y)in A,zt(y,−t)=v0(y,t)in Λ0×(0,φ2),?(y,0)=?0(y),y∈Λ0,

where ∅(zt):=∅1ne1(zt)+∫φ1φ2∅2(j)H∗2(z∗t(t−j))dj.

Here, A⊂RM(M≥1) is a bounded domain with a smooth boundary ∂A=Λ1∪Λ0, where Λ1 and Λ0 are disjoint, closed subsets. The functions P, Z, F, and N represent the memory kernel, the boundary feedback, and the acoustic boundary conditions, respectively. The term involving ∅(zt) represents the nonlinear distributed delay feedback. To handle the distributed delay, we introduce an auxiliary variable η(y,ν,j,t)=zt(y,t−jν) for (y,ν,j,t)∈D=Λ1×(0,1)×(φ1,φ2)×R+, which satisfies the transport equation:

{jηt(y,ν,j,t)+ην(y,ν,j,t)=0,η(y,0,j,t)=zt(y,t).

This transforms the original problem with delay into a system of partial differential equations without explicit delay, which is more amenable to analysis.

2.2. Participants/Sample

This study is purely theoretical and does not involve human or animal participants. The "participants" or "samples" in this context are the mathematical objects of study, namely the solutions to the system of partial differential equations. The initial data for the problem, z0,z1,v0, and ∅0, are assumed to belong to appropriate function spaces that ensure the existence of a weak solution. Specifically, we assume that z0,z1∈PΛ01(A)∩B2(Γ), v0∈B2(Λ0×(0,1)×(φ1,φ2)), and ∅0∈B2(Λ0).

2.3. Materials and Apparatus

The materials and apparatus for this research are the mathematical tools and techniques of modern analysis. These include:

- **Functional Analysis:** The theory of Hilbert spaces, Sobolev spaces, and the properties of linear and nonlinear operators.
- **Partial Differential Equations:** The theory of existence, uniqueness, and regularity of solutions to hyperbolic and parabolic equations.
- **The Multiplier Method:** A powerful technique for obtaining energy estimates for solutions to partial differential equations. This involves multiplying the equation by a suitable function (the "multiplier") and integrating by parts.
- **Lyapunov's Direct Method:** A method for proving the stability of dynamical systems by constructing a scalar function (a "Lyapunov function") whose properties can be used to infer the stability of the system.
- **Convex Analysis:** The properties of convex functions and their conjugates, including Jensen's inequality and Young's inequality, are used to handle the nonlinear terms in the system.

2.4. Experimental Procedure/Data Collection Protocol

The "experimental procedure" in this theoretical study consists of a rigorous mathematical proof. The steps are as follows:

1. **Problem Formulation:** The physical problem is translated into a well-defined mathematical model, as described in Section 2.1.
2. **Well-Posedness:** The existence of a unique weak solution is established using the Faedo-Galerkin method, combined with results from previous studies [37, 38, 39]. This ensures that the problem is mathematically sound and has a solution to be analyzed.
3. **Energy Functional:** An energy functional for the system is defined. This functional represents the total energy of the system, including the kinetic energy, potential energy, and energy stored in the viscoelastic material and at the boundary.
4. **Energy Decay Analysis:** The time derivative of the energy functional is computed, and it is shown that the energy is a non-increasing function of time. This is a crucial step in proving the stability of the system.
5. **Construction of a Lyapunov Functional:** A more general Lyapunov functional is constructed by adding carefully chosen perturbation terms to the energy functional. These perturbation terms are designed to capture the dissipative effects of the memory kernel and the boundary feedback.
6. **Derivation of a Differential Inequality:** The time derivative of the Lyapunov functional is estimated, leading to a differential inequality that relates the Lyapunov functional to its derivative.
7. **Asymptotic Analysis:** The differential inequality is solved to obtain the asymptotic behavior of the Lyapunov functional, and hence the energy functional, as time tends to infinity. This provides the desired decay rate for the energy of the system.

2.5. Data Analysis Plan

The "data" in this study are the mathematical expressions and inequalities obtained during the proof. The analysis of this data involves a series of logical deductions and mathematical manipulations to arrive at the final conclusions. The key steps in the data analysis are:

- **Estimation of Terms:** The various terms in the time derivative of the Lyapunov functional are carefully estimated using a combination of Hölder's inequality, Young's inequality, Poincaré's inequality, and the specific assumptions on the functions in the model.
- **Choice of Parameters:** The analysis involves the introduction of several positive constants that need to be chosen appropriately to ensure that the desired inequalities hold. This often involves a multi-step process of choosing some constants to be sufficiently small and others to be sufficiently large.
- **Case Analysis:** The analysis of the decay rate is divided into two cases, depending on whether the nonlinear function in the boundary feedback is linear or nonlinear in a neighborhood of the origin. This is necessary because the properties of the function in these two cases lead to different types of decay estimates.
- **Use of Convexity:** Jensen's inequality for convex functions is used to handle the nonlinear term in the boundary feedback when the function is nonlinear. This is a key step in obtaining a general decay result that is not limited to specific forms of nonlinearity.
- **Integration of the Differential Inequality:** The final step in the analysis is to integrate the differential inequality for the Lyapunov functional. This integration yields the explicit decay rate for the energy of the system.

3. Results

3.1. Preliminary Analyses

The foundation of our analysis is the establishment of the well-posedness of the problem and the behavior of the energy functional. A crucial first step is to define a suitable energy functional for the system.

Lemma 2.1: Energy Functional
The energy functional E(t) for the system is given by:
E(t)=21||zt||22+21(1−∫0tZ(z)dz)||∇z(t)||22+21∫Λ0 N(y)Z2dΛ+21(Z∘∇z)(t)+∫Λ0∫01∫φ1φ2j|φ2j|φ2 (j)|Z(η(y,ν,j,t))djdνdΛ
The time derivative of this energy functional, E′(t), satisfies the inequality:
E′(t)≤−ϕ1∫Λ0zt//1(zt)dΛ+21(Z′∘∇z)(t)−∫Λ0F(y)Zt2dΛ−21 Z(t)||∇z(t)||22−ϕ2∫Λ0∫φ1φ2|%2(j)|η(y,1,j,t)⍰=2 (η(y,1,j,t))djdΛ≤0
This inequality demonstrates that the energy of the system is non-increasing over time, which is a fundamental property of a dissipative system. Furthermore, we establish the existence of a weak solution to the problem.

Theorem 2.2: Existence of a Weak Solution
Under the assumptions (6)-(11) in the original work, there exists a weak solution (z, eta, emptyset) to the problem (14) for any initial data z0,z1∈PΛ01(A)∩B2(Γ), v0∈B2(Λ0 ×(0,1)×(φ1,φ2)), and ∅0∈B2(Λ0), with appropriate regularity properties. This theorem is proven using the Faedo-Galerkin approach, drawing on established results from the literature [37, 38, 39].

3.2. Main Findings

The central result of this study is the general decay of the energy of the system. To establish this, we introduce a

Lyapunov functional W(t) defined as:
$W(t):=ME(t)+\Omega(t)+PE(t)+\Upsilon(t)$
where M and P are positive constants to be determined, and Ω(t), Ξ(t), and ϒ(t) are auxiliary functionals designed to capture the dissipative effects of the system:
$\Omega(t):=\int Az(t)zt(t)dy+\int \Lambda 0z\odot d\Lambda+21\int \Lambda 0.\forall(y)\oslash 2d\Lambda$
$\Xi(t):=-\int Rzt\int 0tZ(t-z)(z(t)-z(z))dzdy$
$\Upsilon(t):=\int \Lambda 1\int 01\int q1\varphi 2je-\nu j|Z2(j)|.Z(\eta(y,\nu,j,t))djd\nu d\Lambda$
Through a series of technical lemmas (Lemmas 3.1, 3.2, and 3.3), we derive estimates for the time derivatives of these functionals. This culminates in the key result concerning the Lyapunov functional W(t).

Lemma 3.4: Properties of the Lyapunov Functional
There exist positive constants Cj (J=1,...,5) and a time t00 such that the Lyapunov functional W(t) satisfies the following inequalities for all t≥t0:
$W'(t)\le -C1E(t)+C2\int \Lambda 0F12(zt)d\Lambda+C3(Z\circ\nabla z)(t)$
and
$C4E(t)\le W(t)\le C5E(t)$
The second inequality shows that the Lyapunov functional is equivalent to the energy functional, which is essential for relating the decay of W(t) to the decay of the energy. Building on these preliminary results, we present the main theorem on the general decay of the energy.

Theorem 3.5: General Decay of Energy
Let the assumptions (6)-(11) hold. Then there exist positive constants λ1, λ2, a time t0, and ω0∈(0,ω] such that the energy of the system satisfies:
$E(t)\le\lambda 1P-1\{\lambda 2(1+\int t0t\vartheta(\zeta)d\zeta)\},\forall t\ge t0$
where $P(t):=\int t1H(\rho)1d\rho$, and H(t) is a function that depends on whether the function P (related to the nonlinearity) is linear or nonlinear on the interval [0,ω]. The function ϑ(t) is related to the memory kernel and satisfies certain properties ensuring its decay over time. This theorem provides a general decay rate for the energy of the system. The specific form of the decay depends on the properties of the memory kernel (through the function ϑ) and the nonlinearity in the boundary feedback (through the function P).

3.3. Secondary or Exploratory Findings

The proof of the main theorem reveals important insights into the dissipative mechanisms of the system. In particular, the analysis is divided into two cases based on the behavior of the nonlinearity g_1 near the origin.

- **Case 1: P is linear on [0,ω]** In this case, the decay rate is determined by the properties of the memory kernel, as captured by the function ϑ(t). The analysis shows that the energy decays at a rate related to the integral of ϑ(t).
- **Case 2: P is nonlinear on (0,ω]** When the nonlinearity is more complex, the decay rate is also influenced by the function P, which characterizes the nonlinearity. The use of Jensen's inequality is crucial in this case to handle the nonlinear term and obtain a general decay result. The resulting decay rate is given in terms of the inverse of the function P, which highlights the role of the nonlinearity in the energy dissipation process.

These findings demonstrate that the asymptotic behavior of the system is a result of a complex interplay between the viscoelastic damping in the bulk of the material, the acoustic properties of the boundary, and the nonlinear, delayed feedback at the boundary.

4. Discussion

4.1. Interpretation of Key Findings

The results presented in this study provide a comprehensive analysis of the asymptotic behavior of a nonlinear viscoelastic wave equation with acoustic boundary conditions and a nonlinear distributed delay in the boundary feedback. The main finding, encapsulated in Theorem 3.5, is that the energy of the system decays to zero as time goes to infinity, and that the rate of this decay can be characterized in terms of the properties of the memory kernel and the nonlinearities in the system. This result is significant for several reasons. First, it confirms the intuitive physical expectation that the combination of viscoelastic damping and boundary feedback should lead to the dissipation of energy in the system. Second, it provides a rigorous mathematical proof of this expectation for a complex and realistic model that incorporates several important physical effects. The general nature of the decay result, which does not assume a specific form for the memory kernel or the nonlinearities, makes it applicable to a wide range of materials and physical scenarios.

The dependence of the decay rate on the function ϑ(t), which is related to the memory kernel, highlights the crucial role of the material's memory in the dissipation process. A faster decaying memory kernel (corresponding to a faster decaying ϑ(t)) will lead to a slower decay of the energy, as the material "forgets" its past deformations more quickly, reducing the effectiveness of the viscoelastic damping. The influence of the nonlinearity in the boundary feedback, characterized by the function P, is also clearly demonstrated. The distinction between the linear and nonlinear cases shows that the behavior of the feedback at small amplitudes can have a significant impact on the long-term behavior of the system. This is a common feature in the analysis of nonlinear systems, where the local behavior near an equilibrium point often determines the global stability properties.

4.2. Comparison with Previous Literature

The findings of this study are consistent with and extend the results of previous research in the field. Our work can be seen as a generalization of several earlier studies that considered simpler models. For instance, our results are in line with the work of Al-Mahdi and Al-Gharabli [9] and Messaoudi and Al-Gharabli [10], who studied viscoelastic equations with boundary feedback and demonstrated the importance of memory effects in energy dissipation. Our analysis extends their work by considering a more general form of nonlinearity, a distributed delay, and acoustic boundary conditions. The inclusion of a distributed delay in our model connects our work to the extensive literature on the stability of wave equations with delay [14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 34]. While many of these studies have shown that delay can have a destabilizing effect, our results demonstrate that under appropriate conditions, stability can still be achieved in the presence of a distributed delay. This is consistent with the findings of Nicaise and Pignotti [24] for the wave equation with internal or boundary distributed delay.

Our study also builds upon recent work on viscoelastic wave equations with acoustic boundary conditions [1, 2, 13, 17, 21]. The work of Lee and Kang [17], in particular, considered a similar problem with a time-varying delay. Our analysis provides a more general decay result by considering a distributed delay and a more general class of memory kernels. The incorporation of acoustic boundary conditions, following the foundational work of Morse and Ingard [33] and Beale and Rosencrans [32], makes our model more physically relevant for a variety of applications. Compared to the study by Choucha and Ouchenane [23], which investigated a viscoelastic wave equation with distributed delay but without acoustic boundary conditions, our work provides a more complete picture by including the effects of acoustic interactions at the boundary. This is a non-trivial extension, as the acoustic boundary conditions introduce additional terms in the energy functional and require a more careful analysis of the boundary terms.

4.3. Strengths and Limitations of the Study

The main strength of this study is its generality. We have considered a complex and realistic model that incorporates several important physical effects, and we have derived a general decay result under a broad set of assumptions. The use of the multiplier method and the construction of a suitable Lyapunov functional are powerful techniques that can be adapted to analyze other related problems.

However, the study also has some limitations. The analysis is carried out for a weak solution, and we do not investigate the regularity of the solution beyond what is necessary to prove the energy decay. A more detailed analysis of the regularity of the solution would be a valuable extension of this work. Another limitation is that the decay rate obtained is not always explicit. The decay rate is given in terms of the inverse of a function that depends on the memory kernel and the nonlinearity, which may not be easy to compute in practice. Obtaining more explicit decay rates for specific classes of memory kernels and nonlinearities would be a useful direction for future research. Finally, the study is purely theoretical, and it would be interesting to compare our results with experimental data or numerical simulations. This would provide a valuable validation of the mathematical model and the theoretical predictions.

4.4. Implications for Theory and Practice

The results of this study have several important implications for both the theory of partial differential equations and for practical applications in science and engineering. From a theoretical perspective, this work contributes to the development of a general framework for the analysis of nonlinear viscoelastic wave equations with complex boundary conditions. The techniques used in this study can be applied to a wide range of related problems, including those with different types of nonlinearities, more general memory kernels, or other forms of boundary feedback.

From a practical perspective, the results of this study provide valuable insights into the behavior of viscoelastic materials and structures. The understanding of energy dissipation and stability is crucial for the design of materials and devices with specific damping properties. For example, in civil engineering, viscoelastic materials are used to dampen vibrations in buildings and bridges. In the automotive industry, they are used to reduce noise and vibration in vehicles. In all of these applications, a detailed understanding of the material's behavior under dynamic loading is essential. The inclusion of acoustic boundary conditions makes our model particularly relevant for applications involving sound and vibration. For example, in the design of acoustic insulation materials, it is important to understand how sound waves interact with the material and how energy is dissipated at the boundaries. Our results provide a theoretical foundation for the analysis and optimization of such materials.

4.5. Conclusion and Future Research Directions

In this study, we have investigated the asymptotic behavior of a nonlinear viscoelastic wave equation with acoustic boundary conditions and a nonlinear distributed delay in the boundary feedback. We have established a general decay result for the energy of the system, demonstrating that the combination of viscoelastic damping and boundary feedback leads to the stabilization of the system. Our work provides a rigorous mathematical foundation for the analysis of a complex and realistic model of viscoelasticity. The results of this study have important implications for the understanding of energy dissipation and stability in a wide range of physical systems.

There are several promising directions for future research. One direction is to extend the analysis to more general classes of materials and boundary conditions. For example, it would be interesting to consider materials with more complex constitutive laws, such as those with fractional order derivatives, as suggested by the works of Ragusa [3, 7], Guariglia [4, 8], Ortigueira and Coito [5], and Li, Dao, and Guo [6]. Another direction is to investigate the effects of other types of damping mechanisms, such as Balakrishnan-Taylor damping. It would also be interesting to consider the effects of a logarithmic source term, which has been studied in other contexts [11, 21]. Finally, it would be valuable to develop numerical methods for solving the system of equations and to compare the numerical results with the theoretical predictions. This would provide a deeper understanding of the dynamics of the system and would help to validate the mathematical model.

References

[1] Choucha A, Yazid F, Ouchenane D, et al. Qualitative analysis of the asymptotic behavior for a viscoelastic wave equation in the presence of acoustic and fractional conditions combined with nonlinear distributed delay in boundary feedback. Discrete Contin Dyn Syst Ser S. 2024.

[2] Choucha A, Boulaaras S, Jan R, et al. Global existence and decay of a viscoelastic wave equation with logarithmic source under acoustic, fractional, and nonlinear delay conditions. Bound Value Prob. 2024;2024(1):145.

[3] Ragusa MA. Commutators of fractional integral operators on vanishing-Morrey spaces. J Glob Optim. 2008;40(1-3):361–368.

[4] Guariglia E. Fractional calculus, zeta functions and Shannon entropy. Open Math. 2021;19(1):87–100.

[5] Ortigueira MD, Coito F. From differences to derivatives. Fract Calc Appl Anal. 2004;7(4):459.

[6] Li C, Dao X, Guo P. Fractional derivatives in complex planes. Nonlinear Anal Theory Methods Appl. 2009;71(5–6):1857–1869.

[7] Ragusa MA. Parabolic Herz spaces and their applications. Appl Math Lett. 2012;25(10):1270–1273.

[8] Guariglia E, Silvestrov S. Fractional-wavelet analysis of positive definite distributions and wavelets on D'(C). In: Eng. Math. II: Algebraic, Stochastic and Analysis Structures for Networks, Data Classification and Optimization. Springer Int. Publ.; 2016. p. 337–353.

[9] Al-Mahdi AM, Al-Gharabli MM. Energy decay in a viscoelastic equation with past history and boundary feedback. Appl Anal. 2022;101(13):4743–4758.

[10] Messaoudi SA, Al-Gharabli MM. A general decay result of a viscoelastic equation with past history and boundary feedback. Z Angew Math Phys. 2015;66(4):1519–1528.

[11] Al-Gharabli MM, Al-Mahdi AM, Messaoudi SA. Decay results for a viscoelastic problem with nonlinear boundary feedback and logarithmic source term. J Dyn Control Syst. 2022;28:71–89.

[12] Al-Gharabli MM, Al-Mahdi AM, Messaoudi SA. General and optimal decay result for a viscoelastic problem with nonlinear boundary feedback. J Dyn Control Syst. 2019;25(4):551–572.

[13] Choucha A, Boulaaras S, Djafari-Rouhani B, et al. Global existence and general decay for a nonlinear wave equation with acoustic and fractional boundary conditions coupling by source and delay terms. Res Appl Math. 2024;23:100476.

[14] Datko R, Lagnese J, Polis MP. An example on the effect of time delays in boundary feedback stabilization of wave equations. SIAM J Control Optim. 1986;24(1):152–156.

[15] Nicaise S, Pignotti C. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM J Control Optim. 2006;45(5):1561–1585.

[16] Shahrouzi M, Boulaaras S, Jan R. Well-posedness and blow-up of solutions for the p(l)-biharmonic wave equation with singular dissipation and variable-exponent logarithmic source. J Pseudo-Differ Oper Appl. 2025;16(1):18.

[17] Lee MJ, Kang JR. General stability for the viscoelastic wave equation with nonlinear time-varying delay, nonlinear damping and acoustic boundary conditions. Mathematics. 2023;11(22):4593.

[18] Liu W, Zhu B, Li G, et al. General decay for a viscoelastic Kirchhoof equation with Balakrishnan-Taylor damping, dynamic boundary conditions and a time-varying delay term. Evol Equ Control Theory. 2017;6(2):239–260.

[19] Nicaise S, Valein J, Fridman E. Stability of the heat and the wave equation with boundary time-varying delays. Discrete Contin Dyn Syst. 2009;2(3):559–581.

[20] Zhang Z, Huang J, Liu Z, et al. Boundary stabilization of a nonlinear viscoelastic equation with interior time-varying delay and nonlinear dissipative boundary feedback. Abstr Appl Anal. 2014;2014:1–15.

[21] Choucha A, Boulaaras S, Jan R, et al. Asymptotic behavior for a viscoelastic wave equation with acoustic and fractional conditions combined by nonlinear time-varying delay in boundary feedback in the presence of logarithmic source term. Math Meth Appl Sci. 2025.

[22] Choucha A, Boulaaras SM, Ouchenane D, et al. Exponential stability of swelling porous elastic with a viscoelastic damping and distributed delay term. J Funct

Spaces. 2021:2021(1):Article ID 5581634, 1–8.
[23] Choucha A, Ouchenane D. Decay results for a viscoelastic wave equation with distributed delay in boundary feedback. Mathematica. 2023;65(88):43–59.
[24] Nicaise S, Pignotti C. Stabilization of the wave equation with boundary or internal distributed delay. Diff Int Equ. 2008;21(9–10):935–958.
[25] Benaissa A, Benguessoum A, Messaoudi SA. Energy decay of solutions for a wave equation with a constant weak delay and a weak internal feedback. Electron J Qual Theory Differ Equ. 2014;11:1–13.
[26] Nicaise S, Pignotti C. Interior feedback stabilization of wave equation with time dependent delays. Electron J Diff Equ. 2011;41:1–20.
[27] Bland DR. The theory of linear viscoelasticity. Mineola: Courier Dover Publications; 2016.
[28] Cavalcanti MM, Cavalcanti VD, Martinez P. Genarel decay rate estimates for viscoelastic dissipative systems. Nonlinear Anal Theory Methods Appl Ser A. 2008;68(1):177–193.
[29] Coleman BD, Noll W. Foundations of linear viscoelasticity. Rev Mod Phys. 1961;33(2):239.
[30] Lasiecka I, Tataru D. Uniform boundary stabilization of semilinar wave equations with nonlinear boundary damping. Differ Integr Equ. 1993;6:507–533.
[31] Mesloub F, Boulaaras S. General decay for a viscoelastic problem with not necessarily decreasing kernel. J Appl Math Comput. 2018;58(1-2):647–665.
[32] Beale JT, Rosencrans SI. Acoustic boundary conditions. Bull Am Math Soc. 1974;80(6):1276–1278.
[33] Morse PM, Ingard KU. Theoretical acoustics. New York: McGraw-Hill; 1986.
[34] Aounallah R, Choucha A, Boulaaras S, et al. Asymptotic behavior of a viscoelastic wave equation with a delay in internal fractional feedback. Arch Control Sci. 2024;34(2):379–413.
[35] Maryati T, Munoz Rivera J, Poblete V, et al. Asymptotic behavior in a laminated beams due interfacial slip with a boundary dissipation of fractional derivative type. Appl Math Optim. 2021;84(1):85–102.
[36] Arnold VI. Mathematical methods of classical mechanics. 1989. (Grad. Texts Math.). Berlin: Springer.
[37] Al-Gharabli MM, Guesmia A, Messaoudi SA. Existence and general decay results for a viscoelastic plate equation with a logharithmic nonline arity. Commun Pure Appl Anal. 2019;18(1):159–180.
[38] Cavalcanti M, Cavalcanti VD, Prates Filho J, et al. Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping. Differ Integral Equ. 2001;14(1):85–116.
[39] Choucha A, Boulaaras S, Ouchenane D, et al. General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping, logarithmic nonlinearity and distributed delay terms. Math Meth Appl Sci. 2021;44(715):5436–5457.