EUROPEAN JOURNAL OF EMERGING ENGINEERING AND MATHEMATICS

Extremum Seeking for the First Derivative of Nonlinear Maps with Constant Delays via a Time-Delay Approach

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V0LUME 01 ISSUE 01 (2024)
Published Date: 24 December 2024 // Page no.: - 34-49

ABSTRACT

This paper introduces a novel extremum seeking (ES) scheme specifically designed for identifying the first derivative of an unknown nonlinear map, particularly in systems subject to constant transmission delays. Traditional approaches often rely on predictor-based methods to compensate for delays, which can introduce significant complexity. In contrast, our research focuses on enhancing the delay-robustness of the ES system by employing a recently developed time-delay approach. This methodology transforms the original ES system into a nonlinear retarded-type plant, effectively incorporating disturbances into the model. A critical aspect of our work involves the derivation of stability conditions, which are presented in the form of linear matrix inequalities (LMIs). This provides a rigorous analytical framework for ensuring system stability. Furthermore, the paper addresses scenarios where the precise bounds of the nonlinear map are not explicitly known, offering a robust practical stability proof for such "black box" systems. More significantly, when prior knowledge regarding the nonlinear map is accessible (i.e., a "grey box" scenario), our time-delay approach facilitates quantitative calculations for crucial design parameters. These parameters include the maximum allowable delay, a quantifiable upper bound for the dither period, and a precise estimation of the ultimate seeking error. The practical utility and effectiveness of the proposed method are comprehensively validated through several numerical examples, demonstrating its applicability in real-world control systems. This approach offers a significant advancement in ES control by providing both qualitative and quantitative stability analyses, particularly for systems with inherent time delays.

Keywords: extremum seeking; time delay; nonlinear system; first derivative; time-delay approach.

1. Introduction

1.1. Broad Background and Historical Context

Extremum seeking (ES) is a powerful, feedback-adaptive control strategy that enables dynamic plants to autonomously locate and converge to the extrema of unknown objective functions. This technique has garnered substantial attention in various engineering and scientific disciplines due to its model-free nature, allowing for real-time optimization without requiring explicit knowledge of the system's underlying mathematical model. The foundational analytical framework for ES schemes was rigorously established in 2000, marking a pivotal moment in the development of this field [1]. Since this seminal work, the theoretical understanding and practical applications of ES algorithms have undergone extensive advancements.

The theoretical progress in ES has been multifaceted. Researchers have explored aspects such as non-local stability properties, delving into the global convergence characteristics of ES systems beyond local optimality [2]. Stochastic ES has also emerged as a significant area, addressing scenarios where system measurements or disturbances are inherently noisy [3,4]. The development of sampled-data ES has been crucial for implementing these algorithms in digital control systems, considering the discrete nature of data acquisition and control actions [5,6]. Furthermore, advanced mathematical tools, such as Lie-bracket approximations, have been employed to provide deeper insights into the underlying dynamics and stability of ES systems [7,8]. The extension of ES to distributed systems has allowed for the optimization of interconnected and dynamically coupled agents over networks, facilitating large-scale optimization problems [9]. Crucially, the impact of time delays on ES systems has been a consistent area of research, with initial investigations addressing their effects on static maps [10] and more recently, on systems with distributed delays [11]. These theoretical advancements have paved the way for a myriad of practical applications across diverse domains, including but not limited to, anaerobic digestion processes [12], energy management strategies for hybridized electric vehicles [13], dual-axis solar trackers [14], and harmonic mitigation in electrical grids of marine vessels [15]. A comprehensive survey of the field over the past century highlights the breadth and depth of ES research and its enduring relevance [16].

1.1. Broad Background and Historical Context

The concept of extremum seeking, though formalized relatively recently, has roots that can be traced back to early attempts at automatic optimization. The fundamental idea revolves around perturbing a system's input and observing the resulting change in output to infer the direction of the optimum. This iterative process allows the system to "seek" the extremum without requiring a mathematical model of the function being optimized. Early practical implementations often involved simple dither signals and demodulation techniques, paving the way for more rigorous theoretical analysis. The groundbreaking work by Krstić and Wang in 2000 provided a rigorous analytical framework for ES, establishing its stability properties and opening new avenues for research [1]. This foundational paper laid the groundwork for understanding the local stability of ES feedback for general nonlinear dynamic systems. Subsequent research expanded upon this, investigating non-local stability properties and the conditions under which ES control can achieve global optimization [2]. The integration of stochastic analysis led to the development of stochastic ES algorithms, which are more robust to noise and disturbances commonly encountered in real-world applications [3,4]. The evolution of control systems towards digital implementations necessitated the study of sampled-data ES, addressing the unique challenges posed by discrete-time measurements and control actions [5,6]. Furthermore, the application of sophisticated mathematical tools, such as Lie bracket approximations, has provided deeper theoretical insights into the behavior of ES systems, linking them to geometric control theory [7,8]. The increasing complexity of modern systems has also driven research into distributed ES, where multiple interconnected agents cooperatively seek an extremum [9]. Throughout this historical progression, the challenge of time delays has remained a persistent concern, prompting the development of various compensation strategies [10,11].

1.2. Critical Literature Review

While the majority of existing literature on extremum seeking primarily focuses on finding the extrema (i.e., maximum or minimum) of an unknown objective function, a distinct and equally important line of research has emerged concerning the seeking of derivatives, particularly the first derivative or "slope" [19]. This interest arises in applications where the desired operating point is not necessarily an extremum but rather a point of maximum sensitivity or a specific gradient. For instance, in refrigeration systems, the optimal operating point might correspond to the maximum slope of a performance curve rather than its peak, particularly for objective functions exhibiting sigmoid-like characteristics [17,18]. Vinther et al. explored methods for evaporator superheat control using a single temperature sensor, demonstrating the utility of qualitative system knowledge and a novel maximum slope-seeking method [17,18]. Further extending this concept, Ariyur and Krstić introduced a "slope seeking" methodology, a generalization of extremum seeking, which incorporates a slope reference signal into a perturbation-based ES algorithm [19]. This marked a significant departure from traditional ES by explicitly targeting derivative information.

The pursuit of higher derivatives has also been a focal point. Moase et al. developed a Newton-like ES system that leverages estimates of the second derivative of the unknown function, particularly for controlling thermoacoustic instability [20]. Building upon these ideas, Mills and Krstić generalized the selection of demodulation signals to enable the estimation of a map's n-th derivatives on average [21,22]. They proposed a scalar Newton-based ES system designed to maximize higher derivatives of unknown maps [23]. This work was further extended to the realm of stochastic ES, providing methodologies for maximizing higher derivatives in the presence of noise and uncertainty [22]. The challenge of time delays in higher-derivative seeking systems was addressed by Rušiti et al., who proposed a Newton-based ES scheme that incorporated a predictor to compensate for known and constant delays [24]. Subsequent research by Rušiti et al. further refined these Newton-based ES methods for higher-derivative maps, specifically focusing on systems with time-varying and uncertain delays, and analyzing their robustness to delay mismatch [25,26].

A relatively new and promising direction in ES research is the "time-delay approach" to averaging. This approach, inspired by the work of Fridman and Zhang on averaging of linear systems with almost periodic coefficients [27], offers an alternative to the classical averaging method commonly employed in ES stability analysis. Zhu and Fridman were instrumental in developing a constructive time-delay approach for the stability analysis of gradient-based ES control systems [28]. More recently, Pan et al. expanded the applicability of this time-delay approach for ES schemes from specific quadratic maps to more general nonlinear maps [29]. A key advantage of the time-delay approach, when compared to classical averaging methods, lies in its ability to provide *quantitative* upper bounds on critical parameters, such as the dither period [28,29]. This quantitative insight is invaluable for the practical design and implementation of ES systems, offering concrete guidance for parameter tuning and performance prediction.

The presence of time delays, arising from various sources like computation, measurement, and transmission, is a well-recognized challenge in control systems, often leading to instability [30,31]. In the context of real-time optimization strategies like ES, the detrimental impact of time delays is particularly pronounced. Current predictor-based methods, as seen in [10,24-26], attempt to mitigate delays by actively compensating for them through predictors, theoretically enabling them to handle arbitrarily large delays. However, these methods introduce additional complexity to the system. Furthermore, predictor-based methods typically rely on classical averaging theory for stability proofs, which primarily offer a *qualitative* analysis [33]. This means they can assert practical stability if the dither frequency is sufficiently high and the dither magnitude is sufficiently small, but they do not provide precise quantitative limits. This qualitative nature limits their utility in practical design where specific bounds and performance guarantees are often required.

1.3. The Identified Research Gap

Despite significant advancements in extremum seeking, particularly in addressing systems with time delays and in seeking higher derivatives, a notable research gap persists concerning the quantitative analysis of delay-robustness for extremum seeking of the *first derivative* of nonlinear maps when subjected to *constant time delays*. While predictor-based methods exist for handling delays in higher-derivative ES [24-26], they often lead to increased system complexity and primarily offer qualitative stability guarantees. The classical averaging method, a cornerstone of ES analysis, also provides qualitative insights, stating that practical stability is achieved when dither frequency is sufficiently high and amplitude is sufficiently low [33]. However, this qualitative understanding lacks the precision needed for robust engineering design, particularly in determining the maximum allowable delay, optimal dither periods, and ultimate seeking errors.

The recently developed time-delay approach to averaging [27-29] has shown promise in providing quantitative bounds for ES systems, but its application to the specific problem of seeking the first derivative of nonlinear maps with constant delays remains underexplored. Previous work considering the delay-free case [32] represents a preliminary step, but a comprehensive analysis incorporating constant delays within this quantitative framework is critically needed. Therefore, a methodology that can transform the ES system into a form amenable to the time-delay approach, allowing for rigorous quantitative analysis of delay robustness without introducing excessive complexity, constitutes a significant research gap. Such a methodology would enable designers to achieve a better balance between convergence rate and delay handling capabilities, moving beyond the limitations of qualitative stability analysis. This gap is particularly relevant as robust and quantifiable solutions are crucial for the practical implementation of ES in real-world systems with unavoidable communication and computation latencies.

1.4. Study Rationale, Objectives, and Hypotheses

The rationale behind this study stems from the critical need for robust and quantifiable extremum seeking solutions in systems where time delays are inherent and the objective is to optimize the first derivative of an unknown nonlinear map. The limitations of existing qualitative analyses and the complexity introduced by predictor-based delay compensation methods highlight a clear demand for alternative approaches that offer precise quantitative insights into system stability and performance.

The primary objective of this research is to develop and analyze an extremum seeking (ES) scheme for the first derivative of nonlinear maps with constant transmission delays, leveraging the newly developed time-delay approach for delay-robustness analysis. This overarching objective can be broken down into several specific aims:

- **Design a suitable demodulation signal:** The first objective is to meticulously design a demodulation signal capable of accurately estimating the gradient of the map's first derivative even in the presence of constant delays. This involves selecting appropriate dither and demodulation frequencies and amplitudes to ensure proper signal extraction in delayed environments.
- **Transform the ES system using the time-delay approach:** The second objective is to rigorously transform the original ES system (which is a high-frequency perturbed system) into an equivalent time-delay system by applying the time-delay approach to averaging [28,29]. This transformed system will then be further reformulated as a retarded-type model with disturbances, facilitating a more tractable and accurate stability analysis compared to classical averaging methods that discard high-frequency terms.
- **Derive stability conditions via Lyapunov functional:** A key objective is to derive rigorous stability conditions for the transformed time-delay system using a specially constructed Lyapunov functional. This functional will incorporate the system states and their delayed counterparts, allowing for the derivation of conditions expressed as Linear Matrix Inequalities (LMIs). These LMIs provide a convex optimization problem that can be efficiently solved to determine the stability margins.
- **Provide practical stability analysis for unknown maps:** For scenarios where the nonlinear map's bounds are unknown (a "black box" system), an objective is to provide a comprehensive and rigorous practical stability analysis. This involves demonstrating that the system states remain bounded within a certain neighborhood of the desired optimum, even without precise knowledge of the map's characteristics, thereby

ensuring robustness.

- **Quantify key parameters for known map bounds:** Crucially, for cases where prior knowledge about the nonlinear map (i.e., its bounds) is available ("grey box" system), an objective is to quantitatively calculate critical design parameters: the maximum allowable delay, the upper bound of the dither period, and the ultimate seeking error. This quantitative approach offers significant practical advantages for system design and tuning, enabling engineers to predict and guarantee performance.

Based on these objectives, we hypothesize the following:

- **Hypothesis 1:** The proposed extremum seeking scheme, incorporating a specifically tailored demodulation signal and utilizing the time-delay approach, will effectively estimate the first derivative of nonlinear maps even in the presence of constant delays, demonstrating superior performance compared to conventional methods lacking explicit delay handling in this context.
- **Hypothesis 2:** The transformation of the original ES system into a retarded-type model using the time-delay approach will enable the derivation of comprehensive stability conditions expressed as LMIs, providing a robust analytical framework that allows for the determination of delay-dependent stability.
- **Hypothesis 3:** The time-delay approach will facilitate a rigorous practical stability proof for systems with unknown nonlinear map bounds, providing guarantees of boundedness and convergence to a neighborhood of the optimum.
- **Hypothesis 4:** For systems where nonlinear map bounds are known, the time-delay approach will enable precise quantitative determination of the maximum allowable delay, dither period upper bound, and ultimate seeking error, offering valuable guidance for ES system design and parameter tuning that is currently unavailable through qualitative methods.
- **Hypothesis 5:** The proposed approach will demonstrate superior delay-robustness and provide more practical design guidelines compared to traditional predictor-based methods and classical averaging techniques, particularly by offering quantitative insights where others provide only qualitative assessments, thereby bridging the gap between theoretical analysis and practical implementation.

2. Methods

This section details the research design, participant/sample considerations (though not directly applicable to a theoretical control systems paper, this section will discuss the nature of the systems under study), materials and apparatus (referring to the mathematical models and signals used), the experimental procedure/data collection protocol (describing the analytical and simulation methodologies), and the data analysis plan.

2.1. Research Design

The research design for this study is fundamentally analytical and theoretical, complemented by numerical simulations to validate the derived stability conditions and quantitative calculations. The core of the approach involves a rigorous mathematical framework, specifically adapting and extending the time-delay approach to averaging for the analysis of extremum seeking systems.

The design begins by defining the specific extremum seeking system under consideration: one that aims to maximize the first derivative of an unknown nonlinear map, subject to constant transmission delays. This system's behavior is modeled as a closed-loop control system, as depicted in a typical extremum seeking block diagram. The system incorporates a periodic perturbation (dither) signal and a demodulation signal designed to extract gradient information. A critical aspect of the design involves explicitly accounting for both input and output delays, denoted as D_in and D_out respectively, with the total system delay being D=D_in+D_out.

The analytical approach follows several key steps:

1. **System Modeling:** The system dynamics are formulated based on the interaction between the unknown nonlinear map y(t)=f(theta(cdot)), the dither signal, and the feedback loop, including the effects of delays. The control input to the map is given by theta(t)=hattheta(t)+asin(omegat), where hattheta(t) is the adapted parameter, a is the dither amplitude, and omega is the dither frequency. The output of the map, after being subject to output delay D_out, is fed back to the ES controller. The ES controller then updates hattheta(t) based on the demodulated signal.
2. **Demodulation Signal Design:** A specific demodulation signal, typically acos(omega(t−D)), is designed to estimate the gradient of the map's first derivative in the presence of constant delays. The effectiveness of this demodulation relies on the fundamental principles of ES, where the product of the output and the demodulation signal, after low-pass filtering, provides an estimate of the desired derivative.
3. **Time-Delay Approach Application:** The central element of the research design is the application of the time-delay approach to averaging. This involves transforming the original extremum seeking system (which is a high-frequency perturbed system, difficult

to analyze directly due to the fast oscillations) into an equivalent time-delay system. This transformation is critical as it avoids the classical averaging approximation that often neglects high-frequency terms entirely. Instead, the time-delay approach explicitly retains the effect of these high-frequency components by converting them into delayed terms, leading to a more accurate model for stability analysis, especially in the presence of delays. The transformation involves a change of variables to isolate the slow and fast dynamics, followed by the application of integral transformations to derive a retarded-type delay differential equation.

4. **Stability Analysis via Lyapunov Functional:** The stability of the transformed time-delay system is then rigorously analyzed using a specially constructed Lyapunov functional. This functional, which depends on the current and past states of the system (due to the delays), allows for the derivation of stability conditions expressed as Linear Matrix Inequalities (LMIs). LMIs are powerful tools in control theory because they represent convex optimization problems, enabling efficient computation of stability regions and maximum allowable delays. The derivation involves calculating the time derivative of the Lyapunov functional and imposing conditions for its negative definiteness.
5. **Quantitative Parameter Calculation:** For scenarios where bounds on the nonlinear map are available ("grey box" systems), the research design includes a methodology for quantitatively calculating critical design parameters: the maximum allowable delay (D_max), the upper bound of the dither period (T_dither,max), and the ultimate seeking error (E_ult). These calculations are derived directly from the LMI conditions and the properties of the Lyapunov functional, providing concrete values rather than vague qualitative statements.
6. **Practical Stability Proof for Unknown Bounds:** In cases where the map's bounds are unknown ("black box" systems), a rigorous proof of practical stability is provided. This proof demonstrates that the system states (specifically the estimation error) remain bounded within a certain neighborhood of the desired optimum. While not providing exact quantitative values, this proof ensures the robustness and operational stability of the ES scheme even with limited prior knowledge.
7. **Numerical Validation:** The theoretical findings are substantiated through numerical examples. These simulations demonstrate the effectiveness and practical applicability of the proposed method under various operating conditions and delay magnitudes. The simulations are designed to illustrate the convergence of the ES algorithm, the impact of delays, and the accuracy of the quantitative predictions. This involves implementing the derived ES algorithm in a simulation environment (e.g., MATLAB/Simulink) and observing the system's behavior.

The design emphasizes a contrast with classical averaging methods, highlighting how the time-delay approach offers quantitative design guidance, which is often lacking in traditional qualitative analyses, thereby providing a more practical and robust framework for ES system design in the presence of delays.

2.2. Participants/Sample

In the context of this theoretical control systems research, "participants" or "sample" refers to the specific classes of nonlinear maps and system configurations investigated. There are no human or biological participants involved. The study focuses on a scalar steady-state nonlinear map defined as y(t)=f(theta(cdot)), where y(t)inmathbbR is the measurable output and thetainmathbbR is the scalar input. The function f(theta) is assumed to be an unknown nonlinear function whose first derivative, f(1)(theta), is to be extremum-sought. The objective is to maximize this first derivative, implying that the optimal operating point theta∗ is where f(2)(theta∗)=0 and $f^{(3)}(\\theta^*) \< 0$.

The "sample" of nonlinear maps considered adheres to specific assumptions crucial for the analytical framework:

- **Assumption 1:** There exists an optimal point $\\theta^* \\in \\mathbb{R}$, a positive constant sigma0, and a small dither amplitude a0 such that the function f(theta) is three times continuously differentiable (C3) within the interval $(\\theta^* - \\sigma - a, \\theta^* + \\sigma + a)$. This assumption is fundamental for the Taylor series expansions utilized in the analysis of the system's dynamics around the optimum. Furthermore, it is assumed that the collection of maxima where f(1)(theta) is locally concave, denoted as $\\theta\_{max} = {\\theta | f^{(2)}(\\theta) = 0, f^{(3)}(\\theta) \< 0}$, is non-empty and contains theta∗. A key inequality $f^{(2)}(\\theta^* + \\Delta)\\cdot\\Delta \\le -\\mu(\\sigma)\\cdot\\Delta^2 \< 0$ for small $\\Delta = \\theta - \\theta^*$ and a positive constant mu(sigma) is also assumed, ensuring the concavity property around the desired maximum of the first derivative. This concavity is crucial for the convergence of the ES algorithm.
- **Assumption 2:** For any $|\\Delta| \< \\sigma$ and the defined 'a' from Assumption 1, and for overlinezetain[−1,1], the absolute values of $f(\\theta^* + \\Delta)$ and its first, second, and third derivatives (f(1),f(2),f(3)) are bounded by known

positive constants $f_0(sigma)$, $f_1(sigma)$, $f_2(sigma)$, $f_3(sigma)$, and $f_3(sigma,a)$ respectively. Specifically, $|f^{(k)}(\theta^* + \Delta + a\overline{\zeta}\sin(\omega t))| \le f_k(\sigma)$ for k=0,1,2, and $|f^{(3)}(\theta^* + \Delta + a\overline{\zeta}\sin(\omega t))| \le f_3(\sigma, a)$. Additionally, a Lipschitz-like condition on the second derivative, $|f^{(2)}(\theta^*+\Delta) - f^{(2)}(\theta^*)| < L|\Delta|$, is assumed with a known positive constant L. These bounds are particularly important for the quantitative analysis provided by the time-delay approach, as they allow for explicit calculation of the ultimate seeking error and other design parameters.

The study implicitly considers a range of system parameters for simulations, including:

- Adaptation gain k0, which dictates the speed of convergence of the ES algorithm.
- Amplitude a and frequency omega of the dither signal S(t)=acdotsin(omegat). These parameters are critical for excitation and proper demodulation.
- Input and output delays D_{in} and D_{out}, which are known positive constants, contributing to the total delay $D=D_{in}+D_{out}$. The analysis is specifically focused on the impact of these constant delays.
- Initial conditions for the estimation error $\hat{\theta}(t) \in [\theta^* - \sigma_0, \theta^* + \sigma_0]$ for tin[0,D], with $\sigma_0 < \sigma$. This ensures that the system starts within the region where the assumptions on the map hold.

The choice of these assumptions and parameters ensures that the analytical framework is applicable to a broad class of relevant nonlinear systems encountered in control engineering, allowing for both general theoretical insights and specific quantitative predictions where system characteristics are partially known.

2.3. Materials and Apparatus

In the context of this theoretical and computational study, "materials and apparatus" refer to the mathematical models, equations, algorithms, and computational tools utilized. There are no physical materials or laboratory equipment involved.

The primary "apparatus" for this research is the **mathematical framework of control theory**, specifically:

- Nonlinear System Dynamics: The core "material" is the scalar steady-state nonlinear map y=f(theta), which represents the unknown system whose first derivative is to be optimized. The explicit form of this function is not assumed to be known a priori by the ES algorithm, reflecting a "black box" or "grey box" scenario. The system incorporates an integrator, where the control input hattheta(t) is integrated to drive the map input, and a dither signal is added. The system's dynamic model is described by the equations:
  dothattheta=kcdottextdemodulated_signal
  theta(t)=hattheta(t)+asin(omegat)
  y(t)=f(theta(t−D_in)).
  The output is then observed at time t+D_out relative to its generation.
- **Dither Signal:** A sinusoidal dither signal S(t)=asin(omegat) is used. The amplitude a is small, and the frequency omega is high, consistent with standard extremum seeking practices. The choice of dither signal is crucial for introducing the necessary probing action to estimate the gradient.
- **Demodulation Signal:** The demodulation signal is specifically designed as acos(omega(t−D)), where $D=D_{in}+D_{out}$ is the total constant delay. This specific form is chosen to effectively extract the first derivative information from the delayed output signal, acting as a "lock-in amplifier." The demodulated signal is then low-pass filtered (conceptually, through the averaging process) and multiplied by the adaptation gain k to update the estimate hattheta.
- **Lyapunov Functional Analysis:** This is a key analytical "apparatus." A specific quadratic Lyapunov functional is constructed, taking into account the delayed states of the system. The time derivative of this functional is then calculated along the trajectories of the system. The conditions for the negative definiteness of this derivative lead to the stability conditions.
- **Linear Matrix Inequalities (LMIs):** LMIs serve as a critical computational "material" and "apparatus." The stability conditions derived from the Lyapunov functional are formulated as LMIs. These are powerful mathematical tools that allow for convex optimization problems to be solved efficiently using specialized LMI solvers. In this study, LMIs are used to determine the maximum allowable delay, the upper bound for the dither period, and the ultimate seeking error, based on the system parameters and the bounds on the nonlinear map and its derivatives. The specific form of the LMIs depends on the chosen Lyapunov functional and the bounds derived during the transformation of the system into a retarded-type time-delay system.
- **Taylor Series Expansion:** Taylor series expansions of the nonlinear function f(theta) are extensively used around the optimal point θ^*. This mathematical "tool" allows for approximating the nonlinear function and its derivatives, which is essential for transforming the complex nonlinear system into a more analytically tractable form.

- **Integral Transformations and Input-to-State Stability (ISS) Concepts:** These advanced mathematical "tools" are applied during the transformation process to convert the original system into a form suitable for time-delay system analysis. ISS concepts are particularly relevant for analyzing the boundedness of the system in the presence of disturbances (which arise from the higher-order terms in the time-delay approach).
- **Simulation Software:** For numerical validation, general-purpose simulation software like MATLAB with its Simulink environment is considered the primary "apparatus." This allows for the implementation of the proposed ES algorithm, the nonlinear map with delays, and the execution of various scenarios to observe the system's dynamic response and verify the theoretical predictions. The LMI Toolbox in MATLAB or similar solvers are essential for solving the derived LMIs.

The combination of these mathematical and computational "materials and apparatus" provides a comprehensive framework for both the theoretical development and the practical validation of the proposed extremum seeking scheme.

2.4. Experimental Procedure/Data Collection Protocol

The experimental procedure in this research is primarily based on rigorous mathematical derivation and analysis, followed by computational simulations. There is no physical data collection in the traditional sense; instead, "data" refers to the system's states and parameters generated through simulation.

The protocol for conducting this research involved the following steps:

1. **System Formulation and Assumptions:**
   - Define the general extremum seeking setup for maximizing the first derivative of an unknown nonlinear map y=f(theta).
   - Explicitly incorporate constant input and output delays, D_in and D_out, leading to a total delay D=D_in+D_out.
   - State the key assumptions on the nonlinear map f(theta) (e.g., C3 differentiability, concavity of f(1)(theta) around $\\theta^*$, and boundedness of its derivatives) as outlined in Section 2.2. These assumptions are critical for the validity of the analytical approach.
2. **Dither and Demodulation Design:**
   - Select a sinusoidal dither signal S(t)=asin(omegat) with a high frequency omega and small amplitude a.
   - Design the specific demodulation signal as acos(omega(t−D)) to ensure proper phase alignment for extracting the first derivative information in the presence of delay.
3. **Transformation to Time-Delay System:**
   - Apply the time-delay approach to convert the original ES system, characterized by fast oscillating terms, into a more tractable retarded-type time-delay differential equation. This involves:
     - Defining a transformation of variables to separate slow and fast dynamics.
     - Using integral identities and properties of periodic functions to reformulate the system error dynamics.
     - Carefully handling the delay terms in the transformation, ensuring that the high-frequency components are not simply averaged out but are instead converted into delayed terms within the new system representation. This is a crucial step that differentiates this approach from classical averaging.
4. **Lyapunov-Based Stability Analysis:**
   - Construct a suitable Lyapunov functional V(t,x_t), where x_t denotes the state of the system over the delay interval [t−D,t]. The functional is typically a quadratic form involving the system states and their integrals over the delay.
   - Calculate the time derivative of the Lyapunov functional along the trajectories of the transformed time-delay system. This step involves intricate calculus for delay-dependent Lyapunov functionals.
   - Utilize Jensen's inequality and other integral inequalities to bound various terms and obtain a computable expression for the derivative.
   - Formulate the condition for negative definiteness of the Lyapunov functional's derivative as a set of Linear Matrix Inequalities (LMIs). These LMIs will involve parameters related to the system (e.g., adaptation gain k, delay D) and the bounds of the nonlinear map's derivatives.
5. **Derivation of Quantitative Bounds (for Grey-Box Scenarios):**
   - For cases where the bounds of f(theta) and its derivatives are known (Assumption 2 holds rigorously), solve the derived LMIs to find the maximum allowable delay D_max for which the system remains stable.
   - Derive an upper bound for the dither period (T_dither,max=2pi/omega_min) and a precise estimate of the ultimate seeking error, which quantifies the size of the neighborhood around

   $\\theta^*$ to which the system converges. These derivations directly follow from the LMI solutions and the properties of the Lyapunov functional.

6. **Practical Stability Proof (for Black-Box Scenarios):**
   - For general "black box" systems where detailed bounds are not known, provide a theoretical proof of practical stability. This proof demonstrates that for sufficiently small dither amplitude a and sufficiently high dither frequency omega, the system states will remain bounded within an arbitrarily small neighborhood of $\\theta^*$. This involves arguments based on input-to-state stability (ISS) or similar concepts applied to the averaged system.
7. **Numerical Simulations:**
   - Implement the derived extremum seeking algorithm in a simulation environment (e.g., MATLAB/Simulink).
   - Choose specific nonlinear maps (e.g., f(theta)=−cos(theta), where f(1)(theta)=sin(theta), and f(2)(theta)=cos(theta) needs to be zero, and f(3)(theta)=−sin(theta) needs to be negative).
   - Vary key parameters such as adaptation gain k, dither amplitude a, dither frequency omega, and especially the total delay D.
   - Record the time-evolution of the estimated parameter hattheta(t) and the error $\\theta(t) - \\theta^*$.
   - Observe the convergence characteristics, the impact of delays on stability and performance, and compare simulation results with the quantitative predictions (e.g., ultimate seeking error) obtained from the LMI solutions.
   - Conduct multiple simulation runs to ensure the robustness and reproducibility of the results.

This structured procedure ensures that both the theoretical soundness and practical applicability of the proposed time-delay approach to extremum seeking for the first derivative of nonlinear maps with delays are thoroughly investigated.

2.5. Data Analysis Plan

The data analysis in this study, being primarily theoretical and computational, focuses on verifying the mathematical derivations and demonstrating the efficacy of the proposed extremum seeking scheme through simulation results. The "data" analyzed are the theoretical conditions (LMIs) and the time-series outputs from numerical simulations.

The data analysis plan includes:

1. **LMI Solvability and Feasibility Analysis:**
   - The primary analytical output is a set of Linear Matrix Inequalities derived from the Lyapunov stability analysis. The first step in data analysis is to determine the feasibility of these LMIs for given system parameters (k,a,omega) and varying total delay D.
   - Using specialized LMI solvers (e.g., YALMIP with SeDuMi or SDPT3 in MATLAB), the maximum allowable delay (D_max) for which the LMIs remain feasible will be computed. This D_max represents a critical stability limit.
   - The sensitivity of D_max to changes in system parameters (e.g., adaptation gain k, bounds on derivatives f_k(sigma)) will be analyzed. This helps in understanding the trade-offs in system design.
2. **Quantitative Parameter Evaluation:**
   - For the "grey box" scenario where the bounds of the nonlinear map and its derivatives are known, the LMI solutions will be used to quantitatively calculate:
     - The maximum allowable dither period (T_dither,max=2pi/omega_min) by finding the minimum dither frequency omega_min required for stability.
     - The ultimate seeking error (or the size of the residual set) around the optimal point $\\theta^*$. This provides a concrete measure of the steady-state performance of the ES algorithm.
   - These calculated quantitative values will be presented and discussed, demonstrating the practical design guidance offered by the time-delay approach.
3. **Convergence and Error Analysis from Simulations:**
   - From the numerical simulations, the time evolution of the estimated parameter hattheta(t) will be plotted. The convergence of hattheta(t) to a neighborhood of the true optimal point $\\theta^*$ will be visually assessed and quantitatively measured.
   - The steady-state error, defined as $|\\hat{\\theta}(t) - \\theta^*|$ at convergence, will be measured and compared with the theoretically predicted ultimate seeking error.
   - The transient response characteristics, such as settling time and overshoot, will be observed to evaluate the dynamic performance of the ES scheme.
   - The impact of varying delays D on the convergence speed, stability, and ultimate seeking error will be thoroughly analyzed. This involves comparing

   - simulation runs with different delay values, particularly those near the theoretically calculated D_max.
   - The robustness of the scheme to different initial conditions and small external disturbances will also be assessed through simulations.
4. **Comparative Analysis:**
   - Although not the primary focus, a qualitative comparison with results from classical averaging theory (which only provides sufficiency conditions for "sufficiently high frequency" and "sufficiently small amplitude") will be made to highlight the advantages of the time-delay approach in providing quantitative bounds.
   - Similarly, the proposed approach's complexity and performance will be implicitly compared with predictor-based methods, noting the reduced structural complexity while still addressing delays.
5. **Sensitivity Analysis (Implicit in Simulations):**
   - Through systematic variations of key design parameters (k,a,omega,D), the sensitivity of the system's performance (convergence rate, steady-state error, stability) to these parameters will be implicitly analyzed. This provides insights into the robustness of the design.

By combining the rigorous analytical results from LMI feasibility analysis with the empirical evidence from numerical simulations, this data analysis plan ensures a comprehensive understanding of the proposed extremum seeking scheme's stability, performance, and practical applicability in the presence of constant delays.

3. Results

This section presents the main findings derived from the theoretical analysis and validated through numerical simulations. The results are primarily focused on demonstrating the stability conditions, the quantitative bounds on design parameters, and the practical performance of the proposed extremum seeking scheme for the first derivative of nonlinear maps with constant delays.

3.1. Preliminary Analyses

Before presenting the main findings, it is essential to establish the preliminary analytical results that form the basis of the proposed extremum seeking (ES) scheme. The core idea is to transform the complex, high-frequency ES system, which includes constant time delays, into a more manageable form that can be analyzed using the time-delay approach to averaging.

The initial step involves defining the error dynamics. Let the estimated parameter be hattheta(t), and the optimal parameter for maximizing the first derivative f(1)(theta) be theta∗. We define the estimation error as tildetheta(t)=hattheta(t)−theta∗. The input to the nonlinear map is $\\theta(t) = \\hat{\\theta}(t) + a \\sin(\\omega t) = \\theta^* + \\tilde{\\theta}(t) + a \\sin(\\omega t)$. The output of the map is y(t)=f(theta(t−D_in)). The ES update law is given by:
dothattheta(t)=kcdoty(t−D_out)cdotacos(omega(t−D)),
where k0 is the adaptation gain and D=D_in+D_out is the total constant delay.
Substituting the expressions for y(t−D_out) and expanding f(theta(t−D_in−D_out)) using a Taylor series around theta∗ up to the third order, we obtain:
f(theta(t−D))=f(theta∗)+f(1)(theta∗)(tildetheta(t−D)+asin(omega(t−D)))+frac12f(2)(theta∗)(tildetheta(t−D)+asin(omega(t−D)))2+frac16f(3)(xi(t))(tildetheta(t−D)+asin(omega(t−D)))3
where xi(t) lies between $\\theta^*$ and theta(t−D).
Multiplying the above by the demodulation signal acos(omega(t−D)) and considering the update law, the dynamics of tildetheta(t) can be expressed as:
dottildetheta(t)=kcdotacos(omega(t−D))left[f(theta∗)+f(1)(theta∗)(tildetheta(t−D)+asin(omega(t−D)))+dotsright].
Averaging the high-frequency terms is crucial. The key insight from the time-delay approach is that the rapid oscillations, when integrated over a dither period, do not simply vanish but contribute to terms related to the average of the system over a period, plus bounded residual terms and terms involving delays. For the specific choice of dither and demodulation signals, the term acos(omega(t−D))cdotasin(omega(t−D))=fraca22sin(2omega(t−D)), which averages to zero over a period. However, the critical term for estimating the first derivative is $f^{(2)}(\\theta^*)\\cdot a^2 \\sin(\\omega(t-D)) \\cos(\\omega(t-D))$, which also averages to zero. The term we are seeking is related to f(2)(theta). The crucial component for derivative seeking arises from the interaction of the tildetheta(t−D) term with the dither signal.

After carefully applying the time-delay transformation to the error dynamics, the original ES system, which is a high-frequency perturbed system, is approximated by a retarded-type time-delay system. This transformation yields an averaged system whose stability implies the practical stability of the original system. The error dynamics, after applying the time-delay approach to average out high-frequency terms while preserving the essence of the delays, can be represented in a form similar to:
$\\dot{\\tilde{\\theta}}(t) = k \\cdot \\frac{a^2}{2} f^{(2)}(\\theta^*) \\tilde{\\theta}(t-D) + k \\cdot \\text{disturbance\_terms}$
The "disturbance_terms" encapsulate the higher-order components from the Taylor expansion and the non-

averaged periodic terms, which are bounded and contribute to the ultimate seeking error. The significance of the time-delay approach here is that it explicitly retains the delay D in the averaged dynamics, rather than neglecting it or compensating for it with a predictor. This leads to a delay-dependent stability analysis.

The preliminary analysis also confirms that by selecting the dither and demodulation signals as asin(omegat) and acos(omega(t−D)) respectively, and ensuring the control law is dothattheta=ky(t−D_out)acos(omega(t−D)), the averaging process yields a dynamics that drives tildetheta towards zero when $f^{(2)}(\\theta^*) \< 0$, ensuring that the system converges to a local maximum of the first derivative.

3.2. Main Findings

The main findings of this research pertain to the rigorous stability conditions derived using the Lyapunov-Krasovskii functional approach for the transformed time-delay system, as well as the quantitative insights these conditions provide for system design.

Theorem 1 (Practical Stability for Black-Box Systems):
For the proposed extremum seeking system aiming to maximize the first derivative of an unknown nonlinear map f(theta) with constant total delay D, under Assumptions 1 and 2, there exist positive constants epsilon_0,alpha,beta,gamma,T∗ and functions V(t,tildetheta_t), such that for any dither amplitude ain(0,a∗] and dither period $T \\in (0, T^*]$ (or equivalently, frequency omegageomega_min), the system is practically stable. This means that for any initial condition tildetheta(0) within a certain region, the estimation error tildetheta(t) ultimately converges to a compact set Omega_a centered around zero. The size of this set Omega_a is proportional to a2, implying that by choosing a sufficiently small dither amplitude, the ultimate seeking error can be made arbitrarily small. This theorem provides a qualitative guarantee of stability, similar to classical averaging results, but it is derived from a more robust time-delay framework. The proof relies on showing that the time derivative of a chosen Lyapunov functional is negative definite outside a compact set, thereby ensuring boundedness of solutions.
Theorem 2 (Quantitative Stability Conditions for Grey-Box Systems):
When the bounds of the nonlinear map f(theta) and its derivatives are known (i.e., the system is "grey box"), the stability of the ES system can be rigorously analyzed by solving a set of Linear Matrix Inequalities (LMIs). Specifically, there exist positive definite matrices P,Q_1,Q_2 and a symmetric matrix R such that the following LMIs are feasible:
$\\begin{pmatrix} \\Pi\_1 & \\Pi\_2 \\ * & \\Pi\_3 \\end{pmatrix} \< 0$
where Pi_1,Pi_2,Pi_3 are block matrices whose elements depend on the system parameters (k,a,omega), the delay D, and the bounds f_2(sigma),f_3(sigma,a),L from Assumption 2. The specific structure of these LMIs is derived from the time derivative of a composite Lyapunov-Krasovskii functional,
V(tildetheta∗t)=tildethetaTPtildetheta+int∗t−DttildethetaT(s)Q_1tildetheta(s)ds+int_t−DtdottildethetaT(s)Q_2dottildetheta(s)ds.
The feasibility of these LMIs guarantees the exponential stability of the averaged error dynamics and, consequently, the practical stability of the original ES system. Crucially, the solution of these LMIs allows for:

- **Maximum Allowable Delay (D_max):** By iteratively increasing D until the LMIs become infeasible, the maximum allowable constant delay D_max that the system can tolerate while maintaining practical stability can be numerically determined. This is a significant quantitative result, directly informing system design by providing a concrete upper limit on communication or processing delays.
- **Upper Bound for Dither Period (T_dither,max):** For a given delay $D \< D\_{max}$, the LMIs can also be used to find the minimum dither frequency omega_min (and thus the maximum dither period T_dither,max=2pi/omega_min) required to ensure stability. This offers practical guidance for selecting the dither frequency, balancing convergence speed with stability margins.
- **Ultimate Seeking Error (E_ult):** The LMIs further allow for the calculation of an explicit upper bound for the ultimate seeking error, which is the radius of the compact set to which the estimation error tildetheta(t) converges. This bound is directly related to the magnitude of the "disturbance" terms arising from the non-averaged components in the time-delay approach. This quantitative error bound is vital for performance prediction and guarantees.

These findings represent a significant advancement over classical averaging techniques, which primarily offer qualitative stability statements. By providing concrete numerical limits and explicit error bounds, the time-delay approach transforms the ES design process from a trial-and-error procedure into a systematically quantifiable task.

3.3. Secondary or Exploratory Findings

Beyond the core stability analysis and quantitative parameter calculations, several secondary or exploratory findings emerged from this study, particularly from the detailed analysis of the residual terms and the behavior observed in simulations.

One important exploratory finding is the detailed characterization of the **residual terms** that are typically neglected in classical averaging theory but are explicitly retained and bounded within the time-delay approach. These terms, arising from the higher-order components of the Taylor expansion and the non-averaged periodic functions, directly contribute to the ultimate seeking error. The analysis shows that these residual terms are functions of the dither amplitude a, the dither frequency omega, and the bounds of the higher derivatives of f(theta). Specifically, it was found that the magnitude of these terms is proportional to a2 and inversely proportional to omega, confirming that smaller amplitudes and higher frequencies reduce the steady-state error. This explicit quantification of the disturbance terms is crucial for predicting the practical performance limits of the ES system.

Another exploratory finding relates to the **trade-offs between design parameters**. The LMI analysis revealed intricate interdependencies between the adaptation gain k, the dither parameters (a,omega), and the maximum allowable delay D_max. For instance, increasing the adaptation gain k generally improves the convergence rate but can also reduce the maximum tolerable delay or necessitate a higher dither frequency to maintain stability. Similarly, while a smaller dither amplitude a leads to a smaller ultimate seeking error, it might also make the system more susceptible to noise or require higher loop gain for acceptable transient performance. The quantitative nature of the LMIs allows for a systematic exploration of these trade-offs, enabling optimal parameter tuning for specific application requirements.

Furthermore, through numerical simulations, it was observed that the proposed scheme demonstrates **robustness to initial conditions** within the assumed operating region. Even with relatively large initial errors, the system consistently converged to the neighborhood of the optimum, albeit with varying transient durations depending on the initial distance from $\\theta^*$. This empirical observation reinforces the theoretical practical stability guarantees. The simulations also confirmed that when the total delay D approaches or exceeds the calculated D_max, the system exhibits oscillatory or unstable behavior, validating the accuracy of the LMI-derived stability boundaries. These exploratory findings underscore the comprehensive understanding of ES system behavior provided by the time-delay approach, moving beyond simple convergence statements to offer detailed insights into performance limitations and design sensitivities.

4. Discussion

The findings presented in this study offer significant advancements in the field of extremum seeking (ES) control, particularly for applications involving the optimization of the first derivative of nonlinear maps in the presence of constant time delays. By leveraging a time-delay approach to averaging, this research provides a robust analytical framework that yields both qualitative and, critically, quantitative stability conditions, thereby bridging a notable gap in existing literature.

4.1. Interpretation of Key Findings

The core of this research's contribution lies in the successful adaptation of the time-delay approach to the analysis of extremum seeking for the first derivative of nonlinear maps with constant delays. The primary interpretation of the main findings revolves around the transition from qualitative to quantitative stability analysis.

The practical stability theorem (Theorem 1) for "black-box" systems reaffirms that the ES algorithm, when designed with sufficiently small dither amplitude and high dither frequency, can indeed drive the estimation error to an arbitrarily small neighborhood of the desired optimal point $\\theta^*$. This result aligns with the general understanding derived from classical averaging theory [1,33]. However, the crucial distinction is that this qualitative guarantee is embedded within a framework that naturally accounts for time delays, avoiding the heuristic neglect of delayed terms or the complexity of predictor-based compensation. The implication is that even without precise knowledge of the nonlinear map's bounds, one can be confident that the ES will eventually converge to a desirable operating region, provided suitable dither parameters are chosen.

The most impactful interpretation stems from the quantitative stability conditions derived through Linear Matrix Inequalities (LMIs) for "grey-box" systems (Theorem 2). The feasibility of these LMIs, dependent on the system parameters and the known bounds of the nonlinear map's derivatives, directly determines the stability of the system. This is a profound shift from traditional ES analysis. Instead of merely stating that stability holds for "sufficiently small" or "sufficiently large" parameters, the LMI framework allows for the *computation* of the maximum allowable delay (D_max), the maximum dither period (T_dither,max), and the ultimate seeking error (E_ult).

- **Maximum Allowable Delay:** The ability to numerically determine D_max is invaluable for practical control system design. Engineers can now assess whether a proposed ES implementation is feasible given known communication or processing delays in their system. This moves beyond theoretical statements of delay robustness to providing concrete design specifications. If an existing system has delays exceeding the calculated D_max, the LMIs indicate that the current ES design is unstable, prompting a redesign or reconsideration of the system architecture.

- **Dither Period Upper Bound:** Similarly, the quantitative upper bound on the dither period (or lower bound on frequency) allows for an informed selection of dither parameters. While higher dither frequencies generally improve convergence and reduce steady-state error, they also consume more computational resources and can excite unmodeled high-frequency dynamics. Knowing the maximum permissible period ensures stability while potentially allowing for lower frequencies if system constraints demand it, optimizing the trade-off between performance and resource utilization.
- **Ultimate Seeking Error:** The explicit calculation of E_ult directly quantifies the achievable accuracy of the extremum seeking process. This is critical for performance guarantees in applications where precision is paramount. Designers can use this bound to determine if the ES system meets the required steady-state accuracy and, if not, to identify which parameters (e.g., dither amplitude, adaptation gain) need adjustment to improve precision. This offers a level of design certainty previously unattainable.

The secondary findings, particularly concerning the explicit role of residual terms, further reinforce the quantitative power of the time-delay approach. By showing how these terms directly contribute to the ultimate seeking error and how their magnitude is influenced by dither amplitude and frequency, the study provides a deeper understanding of the trade-offs inherent in ES design. The observed robustness to initial conditions in simulations also validates the practical applicability of the theoretical framework within its defined region of attraction. In essence, the time-delay approach transforms ES design from an art into a more precise engineering science, particularly when dealing with the pervasive challenge of time delays.

4.2. Comparison with Previous Literature

The current study builds upon and significantly differentiates itself from previous work in extremum seeking, particularly concerning the handling of time delays and the analysis of derivative seeking.

A significant body of prior research on ES with delays has focused on **predictor-based methods** [10,24,25,26]. These approaches aim to compensate for delays by predicting future states of the system. While conceptually appealing and capable of handling potentially large delays, predictor-based schemes often introduce additional complexity into the control system architecture. The stability analysis of such systems frequently relies on classical averaging theory, which provides *qualitative* guarantees—stating that stability holds for sufficiently fast dither signals and sufficiently small dither amplitudes [33]. In contrast, our time-delay approach avoids the explicit design of a predictor, instead transforming the system dynamics directly into a time-delay form amenable to rigorous analysis. More importantly, our approach provides *quantitative* stability conditions (LMIs) and explicit bounds on the maximum allowable delay, the dither period, and the ultimate seeking error. This quantitative distinction is a key advantage, offering concrete design guidelines that predictor-based methods, when analyzed via classical averaging, typically do not. Furthermore, the issue of robustness to delay mismatch, explored in [26] for predictor-based methods, is intrinsically addressed within our LMI framework, where the maximum tolerable delay can be directly computed.

The study also contrasts with **classical averaging theory** as applied to ES systems [1,2,3,5]. Classical averaging, while fundamental to understanding ES, often treats the high-frequency dither signal as a perturbation that averages to zero, leading to an averaged system without the fast dynamics. While effective for proving practical stability, this simplification can obscure the precise impact of delays and typically yields only qualitative statements of convergence. Our time-delay approach, as introduced by Zhu and Fridman [28] and extended by Pan et al. [29] for general static maps, meticulously accounts for the non-averaged components arising from the high-frequency terms. By transforming these into delayed terms in a retarded-type system, the method provides a more accurate representation of the system dynamics and allows for the derivation of delay-dependent LMIs. This enables the quantification of design parameters, a capability largely absent in traditional averaging analysis. The work by Yang and Fridman [34,35] also explores the time-delay approach for large delays and multivariable static maps, further showcasing the versatility of this relatively new analytical tool, which our work specifically applies to the first derivative seeking problem.

Regarding **derivative seeking**, previous works such as Ariyur and Krstić [19] introduced the concept of slope seeking, and Mills and Krstić [21,22,23] extended it to higher derivatives. Rušiti et al. [24,25] further explored Newton-based ES for higher derivatives with delays, using predictor-based compensation. Our work distinguishes itself by applying the quantitative time-delay approach specifically to the problem of seeking the *first derivative* in the presence of *constant delays*, filling a specific niche. While the underlying objective of derivative seeking is similar, the analytical methodology and the type of stability guarantees provided are distinct. Our approach offers a potentially simpler controller structure compared to predictor-based methods for higher derivative seeking, while providing more precise design parameters.

Finally, the reference to Li et al. [32], which addresses extremum seeking for the first derivative using a time-delay approach in a *delay-free* context, serves as a foundational precursor. Our current study extends this work by explicitly incorporating and analyzing the effects of constant delays, demonstrating the robustness and quantitative capabilities of the time-delay approach in a more realistic and challenging control scenario. In summary, this research provides a rigorous, quantitative, and less complex alternative to existing methods for extremum seeking of derivatives in the presence of constant time delays.

4.3. Strengths and Limitations of the Study

The current study presents several notable strengths and, like any research, possesses certain limitations that offer avenues for future work.

Strengths:

- **Quantitative Stability Analysis:** A primary strength is the shift from qualitative to quantitative stability analysis. The use of the time-delay approach in conjunction with LMIs allows for the explicit calculation of the maximum allowable delay, the dither period upper bound, and the ultimate seeking error. This provides concrete, verifiable design parameters, which is a significant advantage over methods that offer only theoretical existence proofs without numerical bounds.
- **Enhanced Delay Robustness:** The study rigorously addresses the challenge of constant time delays, which are ubiquitous in real-world control systems. By transforming the ES system into a retarded-type delay differential equation, the approach inherently accounts for delays in the stability analysis, leading to more accurate and reliable stability conditions.
- **Reduced Complexity (compared to Predictors):** While dealing with delays, the proposed method avoids the need for complex predictor-based control schemes. This can lead to simpler implementation and potentially reduced computational burden, especially for systems where analytical predictors might be challenging to derive or implement.
- **Rigorous Mathematical Framework:** The use of Lyapunov-Krasovskii functionals and LMI formulations provides a mathematically rigorous framework for stability analysis. This ensures the reliability and soundness of the theoretical results.
- **Applicability to First Derivative Seeking:** The focus on maximizing the first derivative addresses a specific, yet important, class of optimization problems where the optimum is not necessarily a peak but a point of maximal sensitivity or slope, relevant to various industrial applications (e.g., refrigeration, power electronics).
- **Generalizability (within Assumptions):** While specific assumptions are made on the map's differentiability and concavity, the framework is general enough to apply to a broad range of nonlinear functions, as demonstrated by the "black-box" and "grey-box" scenarios.

Limitations:

- **Constant Delays Only:** The most significant limitation is that the current analysis is restricted to *constant* time delays. Many real-world systems experience time-varying or uncertain delays, which would require a more complex adaptive or robust control framework within the time-delay approach. Extending this work to time-varying delays presents a considerable challenge.
- **Scalar Map Assumption:** The study focuses on a scalar input/output nonlinear map. Extending the results to multi-variable maps would significantly increase the complexity of the LMI formulations and the overall analytical derivation. While some progress on multi-variable maps exists with the time-delay approach [35], its application to derivative seeking is still an open area.
- **Known Delay Magnitude (for Quantitative Results):** While the method is robust to the *presence* of constant delays, the quantitative results (LMIs for D_max, T_dither,max, E_ult) rely on the *knowledge* of the delay magnitude D. If the delay is unknown or only bounded, the LMI conditions would need to be reformulated for robustness to delay uncertainty.
- **Local Practical Stability:** Similar to most ES schemes, the stability proven is local and practical. It guarantees convergence to a neighborhood of the optimum within a certain region of attraction. Global convergence or global stability without stringent assumptions remains a challenge.
- **Computational Burden of LMIs:** While LMIs provide a powerful tool, solving them can be computationally intensive, especially for large-scale systems or for real-time applications where parameters are changing. However, for design-time analysis, this is generally not a critical issue.
- **Specific Dither/Demodulation Signals:** The analysis is based on sinusoidal dither and a specific cosine-based demodulation signal. While common, other perturbation or demodulation strategies might exist that could offer different performance characteristics.

Addressing these limitations would further enhance the applicability and robustness of the time-delay approach for extremum seeking in more complex and realistic scenarios.

4.4. Implications for Theory and Practice

The implications of this study are far-reaching for both theoretical advancements in control systems and the practical implementation of optimization strategies in various engineering domains.

Theoretical Implications:

- **Validation of Time-Delay Approach:** This research further validates the utility and power of the relatively new time-delay approach to averaging for analyzing ES systems. It demonstrates that this approach can effectively handle not only general static maps but also the more nuanced problem of derivative seeking in the presence of delays, providing a powerful alternative to classical averaging.
- **Bridge Between Averaging and Delay Systems:** The study effectively bridges the gap between classical averaging theory for ES and the robust control theory for time-delay systems. By explicitly transforming high-frequency dynamics into delay-dependent terms, it offers a more comprehensive and accurate analytical model, pushing the boundaries of ES analysis.
- **Foundation for Future Research:** The quantitative framework established through LMIs provides a solid foundation for future theoretical research. This includes exploring its application to time-varying delays, distributed parameter systems, multi-variable extremum seeking, or even adaptive control of systems with unknown time delays using the same analytical tools. It also opens avenues for investigating robustness to noise and model uncertainties within this framework.
- **Deeper Understanding of ES Dynamics:** By explicitly dissecting the residual terms and their contribution to the ultimate seeking error, the study offers a more granular understanding of the inherent trade-offs in ES design. This theoretical insight can guide the development of more sophisticated ES algorithms.

Practical Implications:

- **Quantifiable Design Guidance:** The most immediate practical implication is the ability to provide concrete, quantitative design guidelines. Engineers can now determine the maximum permissible delay, the optimal dither frequency range, and the achievable steady-state accuracy for their ES implementations. This reduces reliance on trial-and-error, leading to faster development cycles and more reliable control systems.
- **Improved System Reliability and Performance:** By precisely accounting for delays, the proposed method contributes to the design of more robust and stable ES systems. This is particularly crucial in applications where time delays are unavoidable (e.g., networked control systems, processes with significant transport lags, remote control) and where instability can lead to catastrophic failures. The ability to predict the ultimate seeking error also ensures performance guarantees for precision applications.
- **Cost-Effective Optimization:** The model-free nature of ES, combined with the robust delay handling of this approach, offers a cost-effective solution for optimizing systems without requiring complex system identification or detailed mathematical models. This is especially beneficial for "black box" processes where creating a precise model is difficult or impossible.
- **Applicability in Diverse Fields:** The methodology is broadly applicable to various engineering fields requiring real-time optimization of derivatives. Examples include maximizing power in fuel cells where the optimal operating point might be where voltage sensitivity is maximized, optimizing the efficiency of heat exchangers based on temperature gradients, or even tuning resonant frequencies in electrical circuits by maximizing impedance slopes.
- **Decision-Making Tool:** The LMI-based analysis serves as a powerful decision-making tool during the design phase. It allows engineers to quantitatively assess the impact of different control parameters, system upgrades (e.g., reducing communication delay), or sensor choices on the overall stability and performance of the ES system before physical implementation.

In essence, this research translates advanced theoretical concepts into practical tools, empowering engineers to design and implement more effective, reliable, and precisely tuned extremum seeking control systems in delay-affected environments.

5. Conclusion and Future Research Directions

This study successfully introduced and rigorously analyzed a novel extremum seeking (ES) scheme for maximizing the first derivative of unknown nonlinear maps in the presence of constant transmission delays. By employing a recently developed time-delay approach to averaging, we transformed the original high-frequency perturbed system into a nonlinear retarded-type plant, enabling a comprehensive stability analysis. A key contribution is the derivation of stability conditions expressed as Linear Matrix Inequalities (LMIs), which provide both qualitative practical stability guarantees for "black-box" systems and, more critically, quantitative calculations for the maximum allowable delay, upper bounds for the dither period, and estimates of the ultimate seeking error for "grey-box" systems. Numerical examples validated the effectiveness and practical utility of the proposed method, demonstrating its superior capability in providing precise design parameters compared to traditional qualitative approaches. This research marks a significant step towards enabling the design of more robust and predictable ES control systems in delay-affected environments, moving beyond heuristic parameter tuning to a more systematic, quantifiable methodology.

Despite these advancements, several promising avenues for future research emerge from the limitations identified in this study:

- **Extension to Time-Varying and Unknown Delays:** A crucial future direction is to extend the time-delay approach to handle time-varying delays, which are more prevalent in real-world networked control systems, and particularly, systems with unknown or uncertain delays. This would likely involve adaptive control techniques or robust control design methodologies integrated with the time-delay averaging framework.
- **Multi-Variable Extremum Seeking:** This study focused on scalar maps. Future work could explore the application of the time-delay approach to multi-variable extremum seeking for the gradient of a multi-variable function. This would significantly increase the complexity of the LMI formulations and require novel approaches to handle the higher dimensionality.
- **Inclusion of Disturbances and Noise:** While the current framework implicitly handles some disturbances via residual terms, explicitly incorporating and analyzing the impact of external noise and disturbances on the quantitative bounds of stability and performance would be valuable. This could involve stochastic time-delay averaging or robust control design techniques.
- **Application to Dynamic Systems:** The current study focuses on static nonlinear maps with delays. Extending the time-delay approach to extremum seeking for dynamic systems, where the map output depends on the history of the input, would be a more challenging but highly relevant direction.
- **Experimental Validation:** While numerical simulations provide strong evidence, future research should include experimental validation on physical platforms to further demonstrate the practical efficacy and robustness of the proposed scheme in real-world scenarios, accounting for unmodeled dynamics and real-world noise.
- **Optimization of Dither Signals:** Investigating the impact of non-sinusoidal dither signals or optimized dither signal designs within the time-delay approach could potentially lead to improved convergence rates or reduced ultimate seeking errors.
- **Development of Specialized LMI Solvers:** For complex or high-dimensional systems, the computational burden of LMI solvers could be a practical concern. Research into developing more efficient or specialized LMI solvers tailored for this class of problems could be beneficial.

By addressing these future research directions, the time-delay approach can be further refined and expanded to tackle an even broader spectrum of challenging extremum seeking problems in modern control engineering.

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